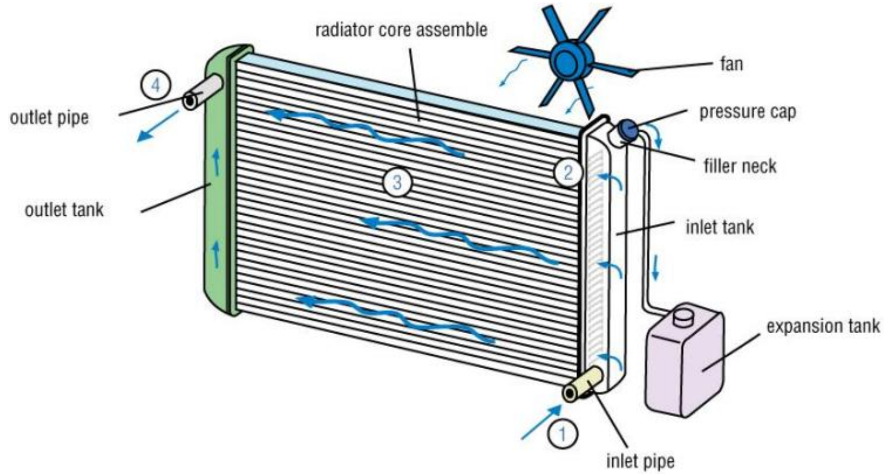
Methods of Heat Transfer



Thermodynamics often makes reference to the heat transfer betwen systems. Often these laws do not adequately describe heat transfer processes, so we must introduce more accurate rules to explain what happens. The control of heat transfer is important to study so that we can design the appropriate tools to transfer thermal energy from one medium to another.

This module introduces heat transfer and the transport laws of conduction, convection and radiation. The laws introduced include Fourier's law, Newton's law of cooling and Stefan-Boltzmann law. Other topics that are discussed include Biot numbers, Wein's law, and the one-dimensional heat diffusion equation. These act as an introduction to the complicated nature of thermal energy transfer.

Methods of Heat Transfer

When a temperature difference is present, heat will flow from hot to cold. Heat can transfer between two mediums by conduction, convection and radiation whenever there is a temperature difference. Recall the first law of thermodynamics. The rate that heat will transfer in a closed system is presented in the following form.

$$Q = W + \frac{dU}{dt}$$

... Eq. (1)

Where Q is the heat transfer rate, W is the work transfer rate and dU/dt is the net change in the total energy of the system. Usually, heat transfer can be analyzed without work being included. However, real systems can include work in their analysis. In the case of only $p \cdot dV$ work occuring, Eq. (1) becomes

$$Q = p \cdot \frac{dV}{dt} + \frac{dU}{dt}$$

... Eq. (2)

with two special cases: constant pressure and constant volume. In the case of constant volume,

$$Q = \frac{dU}{dt} = m \cdot c_v \cdot \frac{dT}{dt}$$

... Eq. (3)

the specific heat capacity is c_v. For constant pressure,

$$Q = \frac{dH}{dt} = m \cdot c_p \cdot \frac{dT}{dt}$$

... Eq. (4)

with the enthalpy $H = U + p \cdot V$ and c_p as the specific heat capacity. The specific heat capacities will be equal in an incompressible liquid, with constant volume at any pressure, rendering $c_v = c_p = c$. The heat transfer rate becomes

$$Q = \frac{dU}{dt} = m \cdot c \cdot \frac{dT}{dt}$$

... Eq. (5)

$U(t)$ is not always known immediately, so most of the time it cannot be used to find Q. To achieve this, we must use the **transport laws** to accurately predict the heat transfer rate. These laws are Fourier's law, Newton's law of cooling, and the Stefan-Boltzmann law introduced in the following sections.

Conduction

Heat Flux and Thermal Conductivity

Conduction is the transfer of thermal energy through the interaction of particles. Small particles transfer kinetic and potential energy as they collide and vibrate with other particles. Two materials can only share energy by conduction if they are in direct or indirect contact with each other. The flow rate of this heat energy is known as **heat flux**.

Heat flux, or thermal flux, is defined as a measurement of the heat rate transfer per unit of area, expressed in watts per square meter ($\frac{W}{m^2}$). Mathematically, it is a vector quantity represented as q.

$$q = \frac{Q}{A}$$

... Eq. (6)

Here, Q is the heat transfer rate and A is the cross-sectional area.

Heat flux from thermal conduction is also proportional to the temperature gradient across an object and opposite in polarity. It varies by a constant *k,* the **thermal conductivity** of a material. The thermal conductivity has units of watts per meter Kelvin ($\frac{W}{m \cdot K}$). It depends on the material and can only be found experimentally. This relationship is known as Fourier's law of heat transfer.

$$q = -k \cdot \frac{dT}{dx}$$

... Eq. (7a)

This is the one-dimensional representation of heat flux.

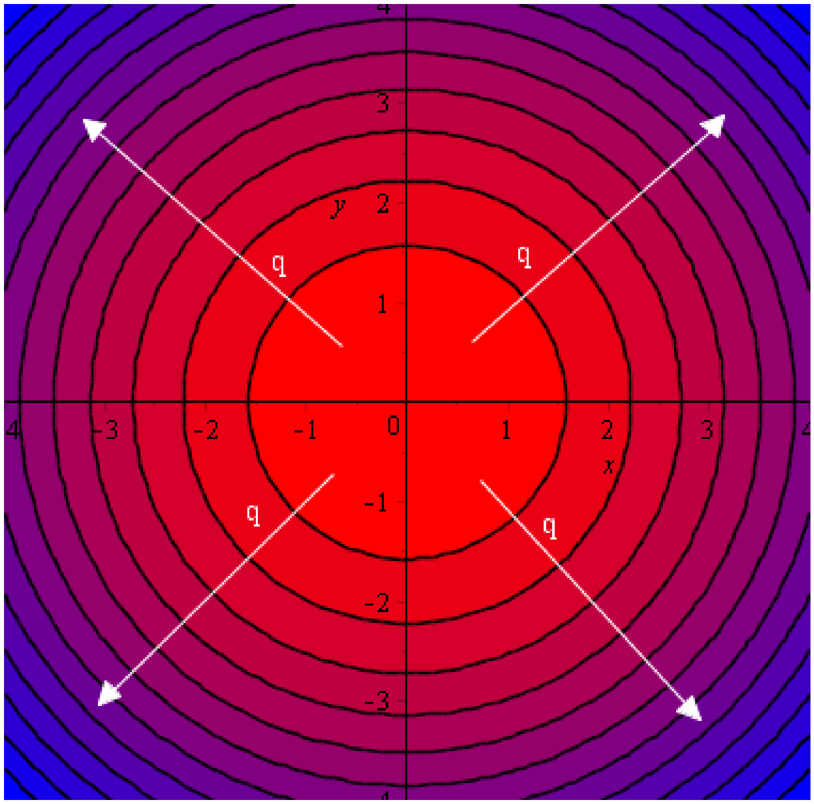


Figure 1: Heat flux shown on a temperature distribution graph.

Because temperature flows from hot to cold, the heat flux will be positive if the rate of change of the temperature gradient decreases. In the multidimensional representation,

$$q = -k \cdot \Delta T = -k \cdot \left(\frac{\partial T}{\partial x} \boldsymbol{i} + \frac{\partial T}{\partial y} \boldsymbol{j} + \frac{\partial T}{\partial z} \boldsymbol{k} \right)$$

... Eq. (7b)

It is sometimes more convenient to work with the scalar form of this equation, such as one dimensional problems where the direction of heat flow is easily determined. Remembering that heat flows from hot to cold, heat flux can be calculated by

$$q = k \cdot \frac{\Delta T}{L}$$

... Eq. (8)

where *L* is the thickness of the material in the direction of the heat flow, and *k* and *T* are positive quantities.

Different materials will conduct heat better than others. Copper, for example, has a very high thermal conductivity because it is a good electrical conductor and can move electrons and transfer energy easily. In contrast, diamond, which is a poor electrical conductor, transfers heat even better than copper due to its efficient lattice structure. Gases are better insulators because the molecules have more space to move around and do not interact as well as solids.

Materials with a high thermal conductivity have a smaller temperature gradient because (from Eq. (7a)),

$$-\frac{dT}{dx} = \frac{q}{k}$$

... Eq. (9)

or more generally,

$$\frac{dT}{dx} \propto \frac{1}{k}$$

... Eq. (10)

For larger thermal conductivities, the temperature gradient will change less.

Heat Flux Example

An industrial furnace is pumping 200 °C hot air through a 3mm copper pipe (k = 400 W / m·K) insulated with 1cm of an elastomer (k = 0.4 W / m·K) on the outside of the pipe. What is the temperature distribution in the copper pipe and the heat conduction through the walls of the tube? Assume an ambient room temperature outside of the tube.

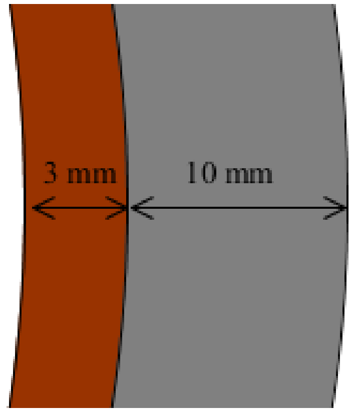


Figure 2: Insulated copper tube

Solution

Examining the given thermal conductivities, we can see that the copper is 1000 times higher than the insulator. The temperature distribution in the copper will be much less than the elastomer here. However, once the heat flux reaches a steady state, the conservation of energy dictates that the heat flux must be equal through both materials. We can equate the two rates to obtain the temperature distribution.

$$q = \left(k \cdot \frac{\Delta T}{L}\right)_{Cu} = \left(k \cdot \frac{\Delta T}{L}\right)_{In}$$

The sum of the heat distribution through the copper pipe and the insulator will be equal to the total change in temperature.

$$(200 - 20)°\mathrm{C} = \Delta T_{Cu} + \Delta T_{In}$$

The temperature gradient of the insulator in terms of the copper is then found from the first equation.

$$\Delta T_{In} = \frac{\left(\frac{k}{L} \cdot \Delta T\right)_{Cu}}{\left(\frac{k}{L}\right)_{In}}$$

Substitute this into the second equation.

$$180 = \Delta T_{Cu} \cdot \left[1 + \frac{(k\ /L)_{Cu}}{(k\ /L)_{In}}\right]$$

$$T_{Cu} := \frac{180}{\left(1 + \frac{\left(\frac{400}{0.003}\right)}{\left(\frac{0.4}{0.01}\right)}\right)} = 0.05398380486$$

Solving this yields ΔT_{Cu} = 0.05 °C for the temperature gradient through the copper. Therefore,

$$T_{In} := 180 - T_{Cu} = 179.9460162 \text{ °C}$$

Although the copper pipe is just under 1/3 the width of the insulator, the high thermal conductivity means that the copper conducts heat very efficiently. In contrast, the insulator has a much higher temperature distribution because it is poorer at conducting heat. The heat flux rate for each material will be

$$q := \frac{400 \cdot T_{Cu}}{0.003} = 7197.840646$$

$$q := \frac{0.4 \cdot T_{In}}{0.01} = 7197.840648$$

So the tube will lose 7.2 kW/m^2 of thermal energy.

1-D Heat Diffusion Equation

In the previous example, we were able to obtain the temperature distribution by equating the heat flux through each material. This can only be done once the heat flux reaches a steady state. What can be done if the heat flux is not constant?

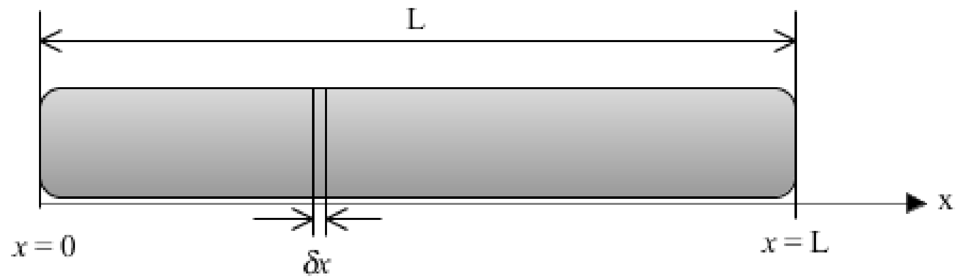


Figure 3: Uniform rod

Consider a metal rod (fig. 3) with the sides insulated and only the ends exposed. This rod has a length *L* that extends from $x = 0$ to $x = L$. This is a uniform rod so assume the specific heat c, density ρ, thermal conductivity k, and cross-sectional area A are constant. The temperature $T(x, t)$ changes with both position x and time t. The amount of heat in the rod at time t is,

$$U(t) = \rho \cdot c \cdot T(x, t) \cdot \delta x$$

... Eq. (11)

Where ρ is the mass per unit length (because we are considering one dimension), and δx is a differential length of the rod. After δt seconds, the amount of heat in the rod will be

$$U(t + \delta t) = \rho \cdot c \cdot T(x, t + \delta t) \cdot \delta x$$

... Eq. (12)

The change in heat will be the difference between these.

$$U(t + \delta t) - U(t) = \rho \cdot c \cdot \left(T(x, t + \delta t) - T(x, t) \right) \cdot \delta x$$

... Eq. (13)

This must be equal to the heat flowing into the rod at x minus the heat flowing out of the rod at $x + \delta x$, for the same duration of time δt. Recall from Eq. (7a) that the heat flow is proportional ot the temperature gradient.

$$U(t + \delta t) - U(t) = \left[\left(-k \cdot \frac{\partial T}{\partial x} \right)_x - \left(-k \cdot \frac{\partial T}{\partial x} \right)_{x + \delta x} \right] \cdot \delta t$$

... Eq. (14)

Equate these two terms and divide by δt and δx.

$$\rho \cdot c \cdot \frac{\left(T(x, t+\delta t) - T(x, t)\right)}{\delta t} = k \cdot \frac{\left(\frac{\partial T}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial T}{\partial x}\right)_{x}}{\delta x}$$

... Eq. (15)

By taking the limit as $\delta x \to 0$ and $\delta t \to 0$, we obtain the one dimensional heat diffusion equation.

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \cdot c} \cdot \frac{\partial^2 T}{\partial x^2}$$

... Eq. (16)

This can be simplified with the **thermal diffusivity** variable $\alpha = \frac{k}{\rho \cdot c}$ in $\left(\frac{m^2}{s}\right)$.

$$\frac{\partial T}{\partial t} = \alpha \cdot \frac{\partial^2 T}{\partial x^2}$$

... Eq. (17)

Convection

Convection occurs when a fluid or gas flows around an object. A small layer of fluid forms around the body, called the **boundary layer**, where heat diffuses from the object to the fluid. The thermal energy is then carried away from the object by the fluid.

Newton's law of cooling states that the temperature difference between the oncoming fluid and the body is proportional to the heat flow from the body. The steady state equation of the law of cooling is written as

$$q = \overline{h} \cdot \left(T_{Body} - T_{Fluid}\right)$$

... Eq. (18)

where q is the heat flow rate, T_{Body} is the temperature of the body and T_{Fluid} is the constant temperature of the oncoming fluid. h is the **film coefficient** or **heat transfer coefficient**, in $\left(\frac{W}{m^2 \cdot K}\right)$. The heat transfer coefficient has 2 forms. h denotes the value a point on the surface while $\overline{h}$ with a bar is the average coefficient over the whole body.

The film coefficient is dependant on the temperature difference only if the fluid is not 'forced' past the body like it is with **forced convection**. Forced convection is method by which fluid flows due to an external source like a fan or pump. With **natural** or **free convection**, h can vary by a degree of the temperature difference such as $\Delta T^{1/2}$ and ΔT^{3}. This happens in situations where the fluid bouys up around a body or when the body is hot enough to boil the fluid, for example. However, our examples will be restricted to situations in which Newton's law of cooling applies or is reasonably accurate.

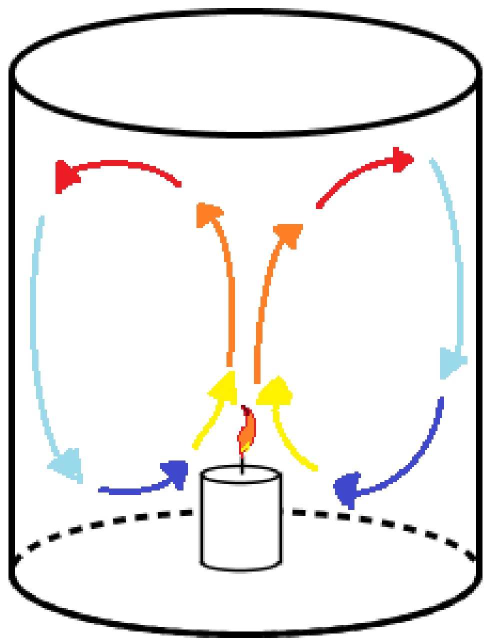


Figure 4: Natural convection. The hot air from the flame expands and rises above the denser cold air.

To predict the transient cooling of convectively cooled objects, we use the **lumped-capacity solution**. Begin by recalling the first law statement,

$$Q = \frac{dU}{dt}$$

... Eq. (5)

By substituting in the equations for Q from Eq. (18) and $\frac{dU}{dt}$ from Eq. (5), this will become

$$-\bar{h}\cdot A\cdot(T-T_{\infty})=\rho\cdot c\cdot V\cdot\frac{d}{dt}(T-T_{Ref})$$

... Eq. (19)

Although T_{∞} represents the temperature of the convection fluid and T_{Ref} is the temperature of U defined at zero, the derivatives of these will both be 0 because they are constants. These differences can be treated as equivalent (assume $T_{Ref}=T_{\infty}$). The solution to this becomes,

$$\ln(T-T_{\infty})=-\frac{t}{\left(\frac{\rho\cdot c\cdot V}{\bar{h}\cdot A}\right)}+C$$

... Eq. (20)

The constant *C* can be found at time *t=0*.

$$C=\ln(T_i-T_{\infty})$$

... Eq. (21)

Substituting this in and rearranging solves for the cooling of the body. This is the lumped-capacity solution.

$$\frac{T-T_{\infty}}{T_i-T_{\infty}}=e^{-\frac{t}{\tau}}$$

... Eq. (22)

Where τ is the **time constant**, which groups together the material properties in Eq. (20), given by

$$\tau=\frac{\rho\cdot c\cdot V}{\bar{h}\cdot A}$$

... Eq. (23)

The thermal conductivity of the material is not included in this constant. It is assumed that heating throughout the object is uniform (with a high thermal conductivity). In these cases, the **Biot Number** for a body is less than 0.1. For other cases, more parameters must be considered.

Biot Number

The Biot number is defined as

$$Bi = \frac{\overline{h} \cdot L}{k}$$

.. .Eq. (24)

The Biot number represents a ratio between the rate that a body cools via conduction compared to convective cooling. It's a unitless quanitity found by equating Fourier's law, Eq. (8), to Newton's law of cooling, Eq. (18).

$$\frac{k}{L} \cdot (T_1 - T_s) = q = \overline{h} \cdot (T_s - T_\infty)$$

... Eq. (25)

Taking the ratio between the temperature differences reveals the Biot number. Fig. 5 displays two different materials that are cooled by convection. The temperature across each material is shown.

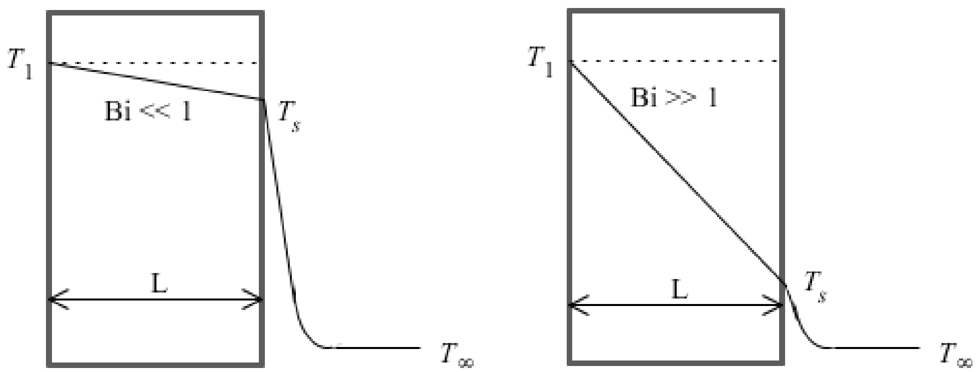


Figure 5: Low and high Biot numbers

In fig. 5, the left object is said to have a low Biot number. It has a low temperature drop within the body $(T_1 - T_s)$ due to a larger thermal conductivity, opposite the second body with a low conductivity. Clearly, the Biot number should be expected to be low because there is a larger drop between the surface temperature and the ambient fluid temperature $(T_s - T_\infty)$. Rearrange Eq. (25) to see this explicitly.

$$Bi = \frac{\overline{h} \cdot L}{k} = \frac{(T_1 - T_s)}{(T_s - T_\infty)}$$

.. Eq. (26)

Radiation

All bodies constantly emit some thermal energy by radiating heat, and energy can travel between bodies in the form of radiation. Thermal energy can radiate across a range of wavelengths but typically it is close to that of infrared. Often, the energy emitted by radiation can be neglected in the presence of conduction and convection at low temperatures. However, at high temperatures, radiation must be considered because the energy emission from a body varies as the fourth degree of the absolute temperature.

As radiation strikes an object, some energy may be absorbed, pass through or reflect off of the surface. A **black body** is an ideal object that does not reflect radiation or let energy pass through. It absorbs all incident radiation and re-emits thermal energy at a rate dependent on the black body, not the incident radiation that heats it.

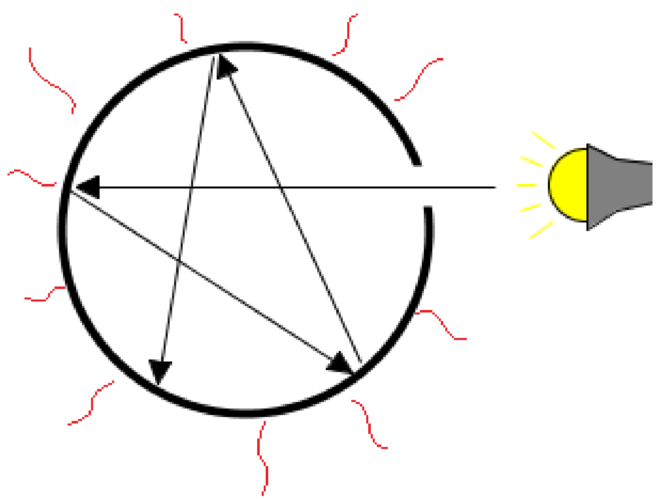


Figure 6: A black body as a cavity with a hole

A popular model of a black body is that of a cavity with a hole through which radiation enters, as shown in figure 6. The walls of the cavity perfectly absorb the radiation and the surface of the body emits thermal energy. The rate that the body emits energy approaches a theoretical maximum, given by the **Stefan-Boltzmann law**.

Stefan-Boltzmann law applies only to black bodies. It states the directly proportional relationship between the energy emitted from a unit area in one second, $e(T)$, to the fourth power of the temperature.

$$e(T) = \sigma \cdot T^4$$

... Eq. (27)

where σ is the Stefan-Boltzmann constant, $\sigma = 5.6704 \cdot 10^{-8} \frac{[\![W]\!]}{[\![m^2]\!][\![K^4]\!]}$. Here, $e(T)$ may be reffered to as the heat flux density, irradiance, or emmisive power, and it is given in watts per square meter. Stefan formulated this in 1879 through experimentation.

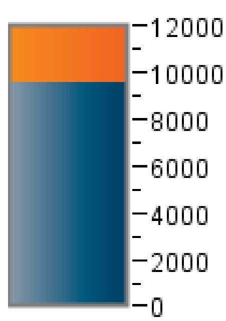
The radiation that a black body will give off will of course be proportional to its temperature. This radiation is given off as a distribution of energy across various wavelengths. For example, a fire iron left in a fire will glow a dull red, emitting its energy mostly through infrared light and a little energy in the visible spectrum. If it were to be heated to a hotter temperature, it would emit more visible light. **Wein's law** declares that the hotter an object is, the shorter the wavelength it will emit through radiation. Wein's law also states that the peak wavelength that radiating energy is emitted at is proportional to its temperature.

$$\lambda_{\max} \cdot T = 2.898 \cdot 10^{-3} [\![m]\!] \cdot [\![K]\!]$$

... Eq. (28)

The constant of proportionality used is referred to as **Wein's displacement constant**. It should be noted that this relationship does not correspond with the peak *frequency* of the radiation through $v = f \cdot \lambda$, that is the peak wavelength $\lambda_{\max}$ and peak frequency $f_{\max}$ are not directly related to each other. Adjust the temperature gauge in fig. 7 and observe its effect on the wavelength's distribution.

Surface Temperature (K)



Peak wavelength λ: 0.3035888323e-6 [meters]

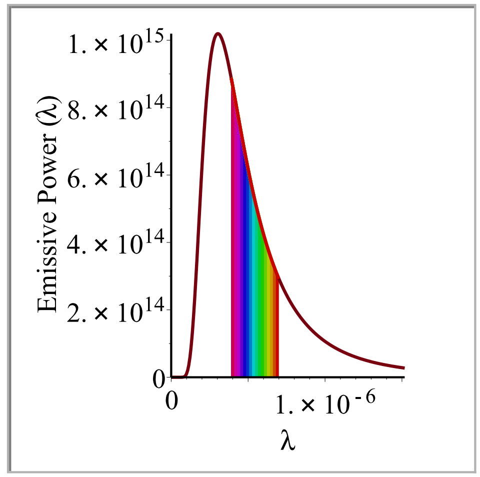


Figure 7: Emissive power vs wavelength chart with adjustable temperature.

The above figure shows the wavelength distribution of thermal radiation vs. the emissive power for a black body and displays the range of visible colour that the body will emit. The temperature gauge sets the surface temperature of the black body. The emissive power depending on lambda was proposed by Max Planck in 1901 and is stated to be

$$e_\lambda = \frac{2\cdot\pi\cdot h\cdot c_0^2}{\lambda^5\cdot\left(\exp\left(\frac{h\cdot c_0}{k_B\cdot T\cdot\lambda}\right)-1\right)}$$

... Eq. (29)

where h is Planck's constant $6.626\cdot 10^{-34}$ $[\![J]\!]\cdot[\![s]\!]$, c_0 is the speed of light $3.00\cdot 10^{8}\ \frac{[\![m]\!]}{[\![s]\!]}$ and k_B is Boltzmann's constant $1.38\cdot 10^{-23}\ \frac{[\![J]\!]}{[\![K]\!]}$.

The net energy that an object can transfer by radiation Q_{net} is the difference between $Q_{1\rightarrow 2}$. If some object radiated heat towards another object, assuming they are both black bodies, the net heat transferred is

$$Q_{net} = \sigma\cdot A_1\cdot\left(T_1^4 - T_2^4\right) = A_1\cdot\left[e_1(T) - e_2(T)\right]$$

... Eq. (30)

This net transfer assumes that all radiating energy from object 1 *sees* the second object. In cases where there are multiple objects, or some radiation diffuses away, this quantity is multiplied by a **view factor** (otherwise referred to as a shape/configuration factor). This is the fraction of energy intercepted by object 2 from object 1.

$$Q_{net} = F_{1\rightarrow 2} \cdot \sigma \cdot A_1 \cdot \left(T_1^4 - T_2^4\right)$$

... Eq. (31)

If the objects are not perfect black bodies, then the view factor is replaced by the **transfer factor** which also accounts for the different surface properties and geometries of the objects.

In some experiments, radiation can obscure a temperature reading. To prevent this we use **radiation shielding** to reduce the amount of radiating thermal energy that strikes the temperature gauge. This can take the form of a reflective sheet, enclosed casing, or multiple layers of similar items that reduce the error of the reading due to thermal radiation.

Examples with MapleSim

Example 1: Conduction Through Two Materials

Example 2: Conduction and Convection

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