Caustics of an Ellipsoid (centro-surface of an Ellipsoid)

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The picture demonstrates one of the most beautiful and mysterious surfaces in mathematics. First studied and drawn by Arthur Cayley and then by many mathematicians after him. The presented picture is obtained using Cartesian coordinates. Probably, it is the first time that cartesian coordinates are used for this purpose. All the previous attempts as far as I know were done using curvilinear coordinates.

Mathematics behind the picture:

Triaxial Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Gaussian Curvature: $K(x,y)=\frac{1}{\left(abc\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)\right)^2}$

Mean Curvature: $H(x,y)=\frac{|x^2+y^2+z^2-a^2-b^2-c^2|}{2(abc)^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{3}{2}}}$

Maximum radii of curvature: $R_1(x,y)=\frac{1}{H-\sqrt{H^2-K}}$

Minimum radii of curvature: $R_2(x,y)=\frac{1}{H+\sqrt{H^2-K}}$

Unit normal vector of the ellipsoid: $N(x,y)=\left(\frac{x}{a^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{1}{2}}},\frac{y}{b^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{1}{2}}},\frac{z}{c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{1}{2}}}\right)$

Focal surface of maximum curvature: $(x,y,z)-N\cdot R_1$

Focal surface of minimum curvature: $(x,y,z)-N\cdot R_2$

Maple commands:

```
restart;

a := 4.5;

b := 3;

c := 1.5;

f := (x, y) -> c*sqrt(1 - x^2/a^2 - y^2/b^2);

K := (x, y) -> 1/(a^2*b^2*c^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^2);
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H := (x, y) -> 1/2*abs(x^2 + y^2 + f(x, y)^2 - a^2 - b^2 - c^2)/(a^2*b^2*c^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(3/2));
```

```
R_1 := (x, y) -> 1/(H(x, y) - (H(x, y)^2 - K(x, y))^(1/2));
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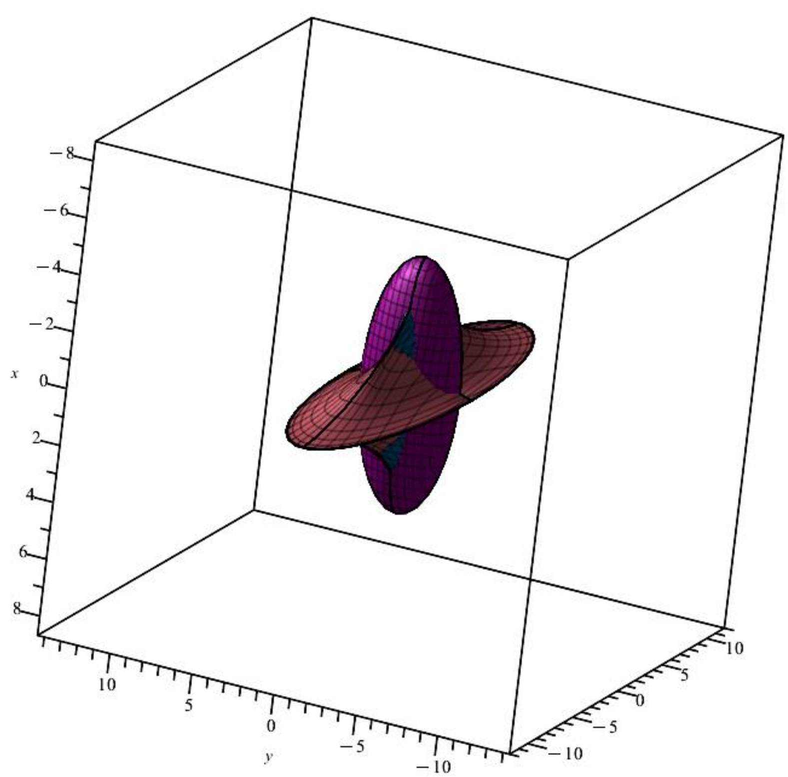
```
R_2 := (x, y) -> 1/(H(x, y) + (H(x, y)^2 - K(x, y))^(1/2));
```

```
plot3d([[f(x, y), -f(x, y)], [a*cos(x), b*sin(x), 0], [a*cos(x), 0, c*sin(x)], [0, b*cos(x), c*sin(x)], [(-a^2 + b^2)*cos(x)^3/a, (-a^2 + b^2)*sin(x)^3/b, 0], [(-a^2 + c^2)*cos(x)^3/a, 0, (-a^2 + c^2)*sin(x)^3/c], [0, (-b^2 + c^2)*cos(x)^3/b, (-b^2 + c^2)*sin(x)^3/c], [0, (-a^2 + b^2)*cos(x)/b, (-a^2 + c^2)*sin(x)/c], [(a^2 - b^2)*cos(x)/a, 0, (-b^2 + c^2)*sin(x)/c], [(a^2 - c^2)*cos(x)/a, (b^2 - c^2)*sin(x)/b, 0], [x - x*R_2(x, y)/(a^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), y - y*R_2(x, y)/(b^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), f(x, y) - f(x, y)*R_2(x, y)/(c^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2))], [x - x*R_2(x, y)/(a^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), y - y*R_2(x, y)/(b^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), -f(x, y) + f(x, y)*R_2(x, y)/(c^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2))], [x - x*R_1(x, y)/(a^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), y - y*R_1(x, y)/(b^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), f(x, y) - f(x, y)*R_1(x, y)/(c^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2))], [x - x*R_1(x, y)/(a^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), y - y*R_1(x, y)/(b^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), -f(x, y) + f(x, y)*R_1(x, y)/(c^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2))]], x = -5 .. 5, y = -5 .. 5, color = [5, 1, 1, 1, 1, 1, 1, 1, 1, 1, 8, 8, 10, 10])
```

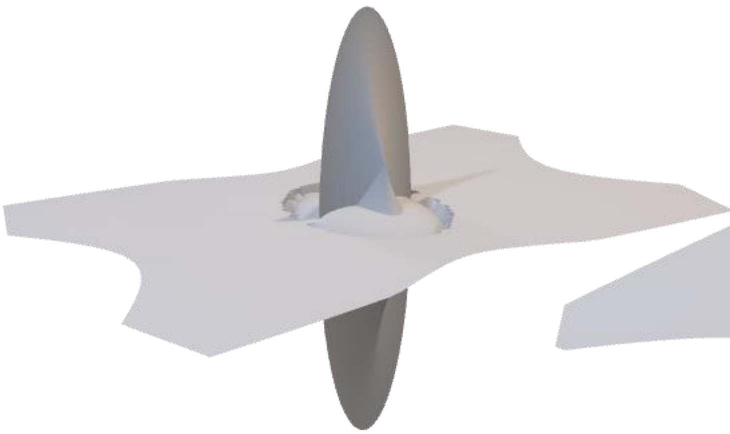
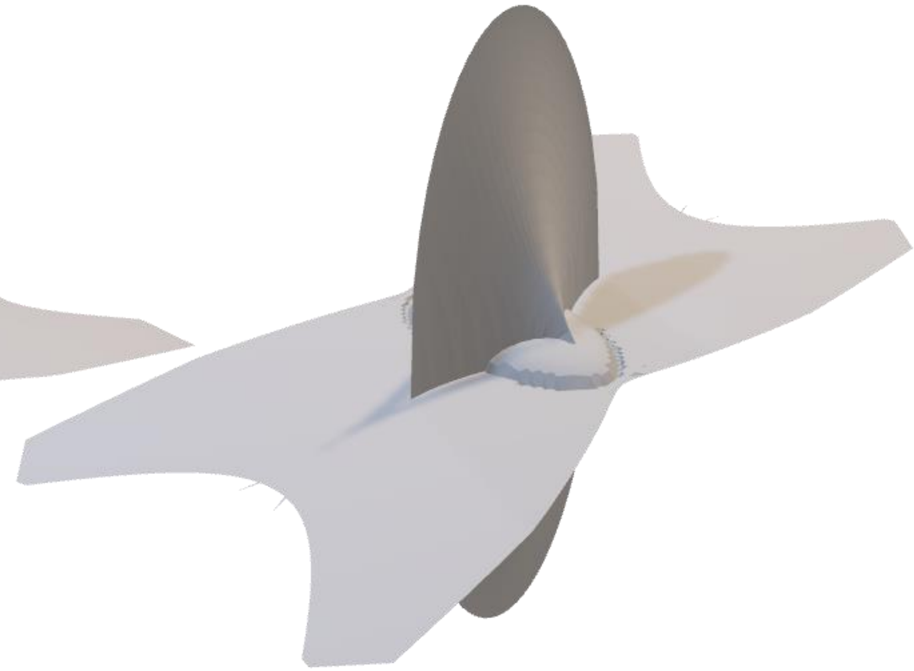
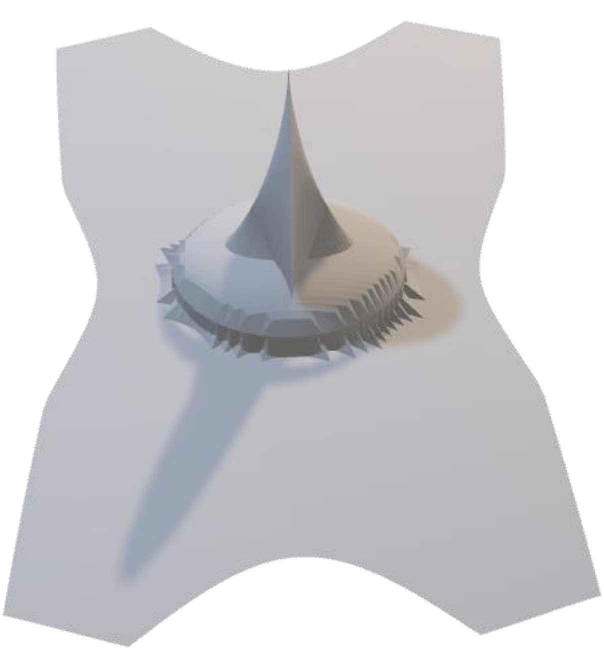
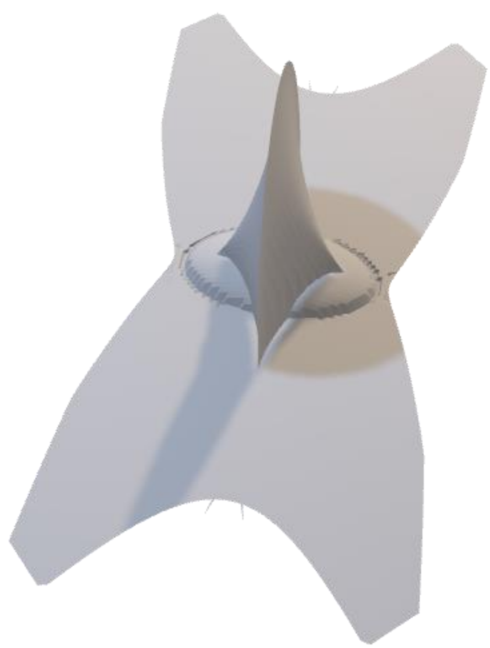
Maple Learn activity:

https://learn.maplesoft.com/d/FPPUAPOJFMJFDQJNBJCQKUNUARLTOFPIDSCGLPMJALNOLNOQJQJGJGBHDFKQIRMOLTFFHTIOPMETGNMLDUFNAJFLDIFPLIFSOTGI



Using export STL file functions of Maple I was able to create some 3D models using the 3d printers in the laboratories of ADA University.

STL files:

Maple code for the creation of STL files:

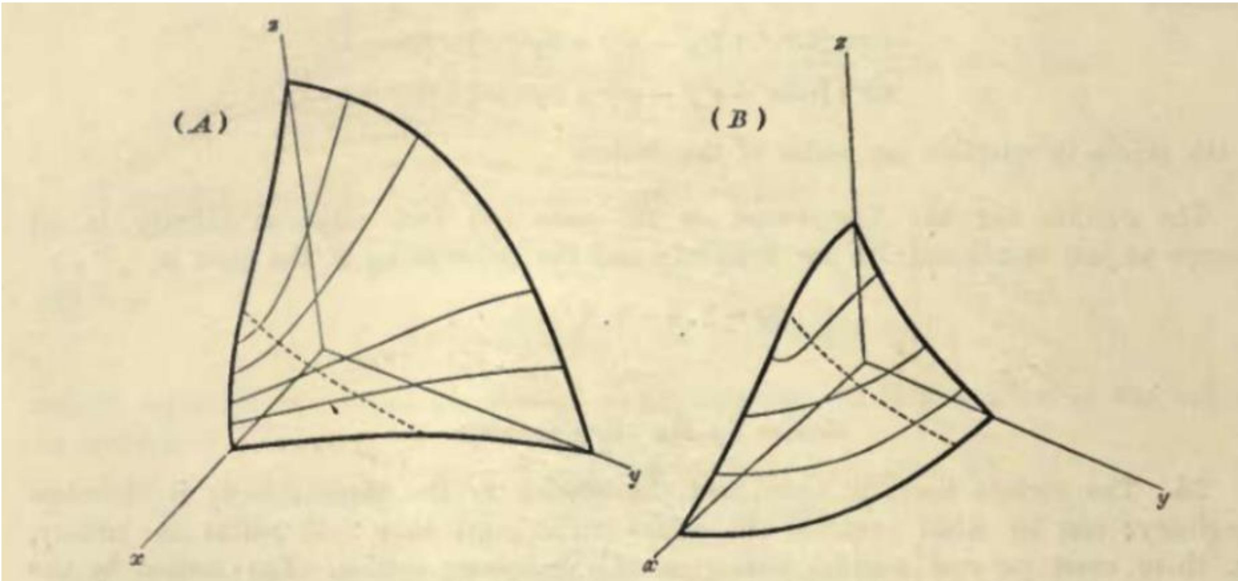
```
caustics := plot3d([[f(x, y), -f(x, y)], [x - x*R_2(x, y)/(a^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), y - y*R_2(x, y)/(b^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), f(x, y) - f(x, y)*R_2(x, y)/(c^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2))], [x - x*R_2(x, y)/(a^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), y - y*R_2(x, y)/(b^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), -f(x, y) + f(x, y)*R_2(x, y)/(c^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2))], [x - x*R_1(x, y)/(a^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), y - y*R_1(x, y)/(b^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), f(x, y) - f(x, y)*R_1(x, y)/(c^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2))], [x - x*R_1(x, y)/(a^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), y - y*R_1(x, y)/(b^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2)), -f(x, y) + f(x, y)*R_1(x, y)/(c^2*(x^2/a^4 + y^2/b^4 + f(x, y)^2/c^4)^(1/2))]], x = -15 .. 15, y = -15 .. 15);

Export("caustics.stl", caustics, base = homedir);

Export("caustics-text.stl", caustics, base = homedir, encoding = text);
```

Physical model (the first picture also shows its support):


The motivation of this Maple Art proposal is to popularize a less known chapter in the history of mathematics: Cayley's attempt to find the shape of the surface containing all the centers of the principal curvatures of an ellipsoid. Cayley called it centro-surface of an ellipsoid. Nowadays, this surface is known as *focal surface* of an ellipsoid, *Cayley's Astroida* or by more popular name (because of applications in Optics and related fields) *caustic* of an ellipsoid. Arthur Cayley's paper from 1873 contains sophisticated algebraic calculations followed by an attempt to draw this surface. He says: "I constructed on a large scale a drawing of the centro-surface for the values $a^2 = 50, b^2 = 25, c^2 = 15$. (These were chosen so that a, b, c should have approximately the integer values 7, 5, 4, and that $a^2 + c^2$ should be well greater than $2b^2$; they give a good form of surface,…"



After A. Cayley there were many attempts to recreate this surface both as drawings and as 3-D models. With the dawn of the computer graphics era the quality and accuracy of these recreations increased and we can now find many interesting and colorful renderings of this surface in books, articles and websites. More pictures and information about history and mathematics can be found in [2].

Acknowledgements

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References

[1] A. Cayley, On the centro-surface of an ellipsoid, Transactions of the Cambridge Philosophical Society, 12(1), 319-365 (1873). Also included in The collected mathematical papers of Arthur Cayley, Vol. VIII, Cambridge University Press, Cambridge, 316-365 (1895). http://name.umdl.umich.edu/ABS3153.0008.001

[2] Yagub N. Aliyev, Apollonius Problem and Caustics of an Ellipsoid, Arxiv, preprint, 2023.

https://arxiv.org/abs/2305.06065