Doubly Companion Matrices

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The entries

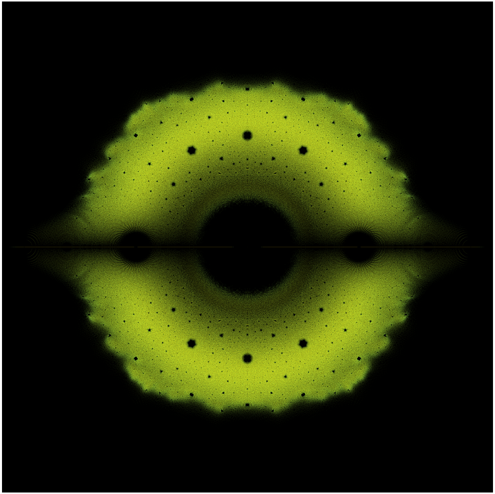

Figure 1: The image for the cover of our new book Computational Discovery on Jupyter, computed in Maple by creating and then solving the 2,184,139 degree 8 characteristic polynomials using `fsolve`.

Persymmetric case

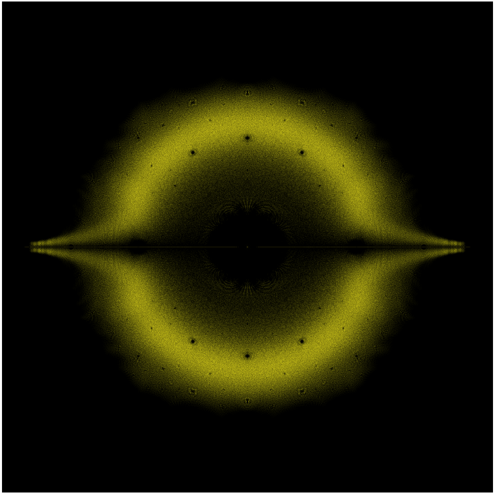

Figure 2: The dimension $m = 12$ *persymmetric* case. No compression here: characteristic polynomials are unique. 6000×6000 grid. Eigenvalue computation by Maple.

Zooming in on the persymmetric edge

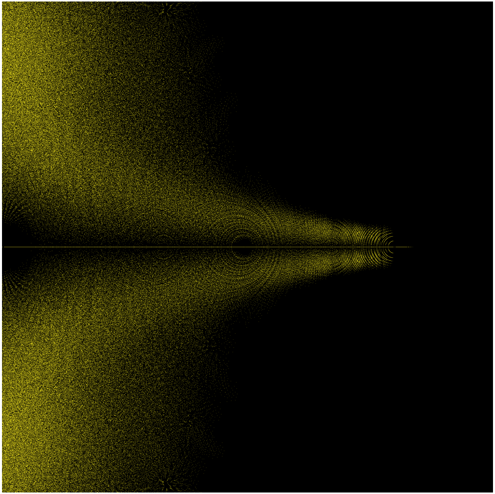

Figure 3: Persymmetric, zoomed to $[1{,}2.25] \times [-0.625{,}0.625]$. Grid is 2000 by 2000. Dimension $m = 12$.

How were the images made?

We use (mostly) the viridis colour palette by Smith and van der Walt, SciPy 2015. John May put this palette into Maple 2022. This is a sequential map, approximately perceptually uniform, and robust under many kinds of colour blindness.

So what do we do, exactly? We choose a grid, and count the eigenvalues in each pixel; we colourize by the chosen palette using an approximate inversion of the frequency distribution (so that roughly “equal areas” are displayed in each colour interval).

I have put one workbook on the Maple Cloud showing exactly how it works.

More on Bohemians

You can find a video on my YouTube channel of a related talk at Bohemian Matrix Geometry.

You can find another related talk at

"Skew Symmetric Tridiagonal Bohemians"

The Maple Transactions papers that last talk refers to are

What can we learn from Bohemian Matrices?
https://doi.org/10.5206/mt.v1i1.14039

and

Skew-symmetric tridiagonal Bohemian matrices
https://doi.org/10.5206/mt.v1i2.14360

See also chapter 5 of our New Book, *Computational Discovery on Jupyter*, with Neil Calkin and Eunice Chan.

The proposed book cover: Doubly Companion Matrices

Link: Version 3

Note in particular the little “rosette” at the origin (which we have put on the spine of the book).

Doubly Companion Matrices

A *doubly companion matrix* is a particular kind of rank-one update to a Frobenius companion matrix:

$$\begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 & -a_5 - b_5 \\ 1 & 0 & 0 & 0 & -b_4 \\ 0 & 1 & 0 & 0 & -b_3 \\ 0 & 0 & 1 & 0 & -b_2 \\ 0 & 0 & 0 & 1 & -b_1 \end{bmatrix} . \qquad (1)$$

"Doubly companion" matrices were introduced in a 1999 paper by Butcher and Chartier in order to improve certain implicit Runge–Kutta methods, and later General Linear Methods, for numerically solving ordinary differential equations. Doubly companion matrices are not studied outside of this application, so far as I know. Rank-one updates, on the other hand...

Some Properties of Doubly Companion (DC) Matrices

- Given the a_k one may choose the b_k so that the spectrum is whatever you please; in particular the charpoly can be λ^m.
- There is an explicit recurrence relation for the characteristic polynomial.
- DC matrices need not be normal
- DC matrices are nonderogatory: Wanicharpichat (2011)

Bohemian DC matrices with population $(-1,1)$

- At dimension m there are 2^{2m} such matrices (the height of the matrix may be 2 if $a_m = b_m$). Up to $m = 8$ we get 4, 16, 64, 256, 1.024, 4.096, 16.384, 65.536 such matrices.
- This set of matrices does *not* include companion matrices for Littlewood polynomials
- Some (2^m) of these matrices are persymmetric

Bohemian DC matrices with population $(-1,0,1)$

- At dimension m there are 3^{2m} such matrices: up to $m = 8$ we get 9, 81, 729, 6.561, 59.049, 531.441, 4.782.969, and 43.046.721 matrices.
- This set of matrices *does* include companion matrices for the Littlewood polynomials; twice, in fact—namely once with all $a_k = 0$ and all $b_k = \pm 1$ and once the other way around.
- Again some (3^m) of these matrices are persymmetric.

The cover image was made from all 43,046,721 DC matrices with population $(-1,0,1)$, using $m = 8$. The image is a colorized density plot of all eigenvalues of this family, on a square $[-1.419,1.419] \times [-1.419 \times 1.419]$ cut into a fine grid, 6000×6000. The eigenvalues were computed using standard methods (NAG Library; LAPACK BLAS).

Zooming in on the origin

If we zoom in on the origin (on the “spine” of the book cover), we see a “rosette” of rounding errors, just as in the skew-symmetric tridiagonal case. This is because some of the matrices are nilpotent, and the highly multiple zero eigenvalue is quite sensitive to rounding errors. Perturbing to $\lambda^8 - 10^{-16}$ gets eigenvalues of magnitude 10^{-2}, which are perfectly visible.

How many matrices have characteristic polynomial λ^8? How many have other (less strong) multiple roots at the origin? How can we find this out?

Exact computation I: Brute Force

By computing *all* 2.184.139 characteristic polynomials of all 43.046.721 matrices using Maple's exact integer arithmetic (this took about 3 hours on my little laptop) we find that the characteristic polynomial λ^8 occurs for exactly 617 of these matrices.

# matrices	nullity
617	8
1,076	7
4,468	6
15,064	5
60,852	4
265,222	3
1,247,698	2
6,425,898	1

Table 1: Matrices with given nullity

Exact computation II: Inverse Bohemian Eigenvalues

J. Rafael Sendra pointed out that we might approach this another way. Let each a_k satisfy $a_k(a_k - 1)(a_k + 1) = 0$, and likewise $b_k(b_k - 1)(b_k + 1) = 0$. Then setting the coefficients of the characteristic polynomial to zero gives 8 more polynomial equations, giving 24 equations in all for these 16 unknowns. We may compute a Gröbner basis for this (rather daunting) system of nonlinear equations, and see if we can count the number of solutions thereby.

Fast Gröbner basis

Computing a total-degree Gröbner basis for these equations takes about 6 seconds on my little laptop. Postprocessing the 452 elements of that basis in order to count the number of solutions (correctly getting 617) takes only milliseconds. Getting the solutions themselves requires building and solving a 617 by 617 eigenvalue problem (using floating-point: we have to round the answers!) which takes more time to set up, but still only about two minutes for each matrix (we need 16 matrices, and then a random linear combination of those 16 in order to avoid spurious multiplicities).

This is the first proof that Rafa's idea can work "in the wild."

Eigenvalues of one of those matrices

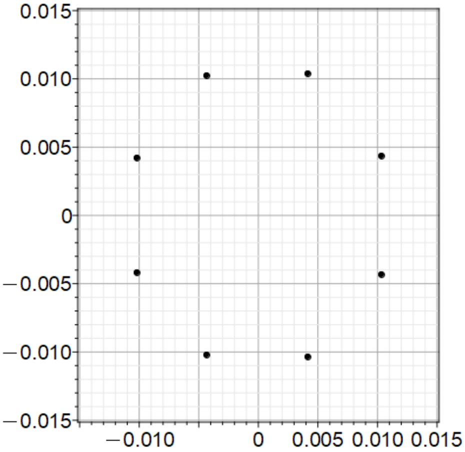

Figure 4: Computed eigenvalues of one of the 617 nilpotent matrices

The matrix in question

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{2}$$

Thank you

Some of these slides were used in a talk at the Bohemian Minisymposium at the meeting of the International Linear Algebra Society in Madrid, Spain, in June 2023.

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