Schwarz-Christoffel triangle

by Neil J Ross
University of St Andrews

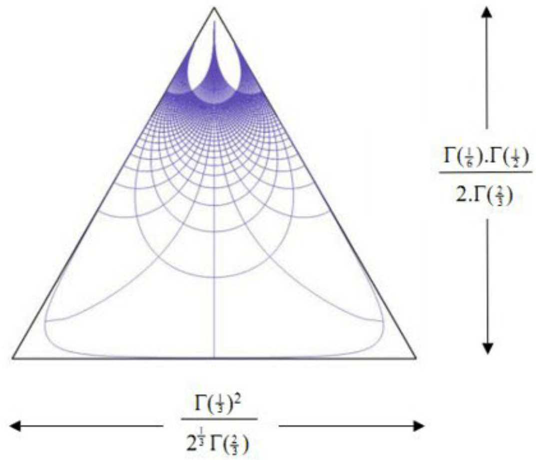
The simplest Schwarz-Christoffel mapping carries the upper complex half plane conformally into the interior of an equilateral triangle with corners at approximately –2.103273 (on the real axis), approximately +2.103273 (on the real axis) and approximately 3.642976i (on the imaginary axis) and is defined by

$$z \longmapsto \int_0^z \frac{d\xi}{(1+\xi)^{\frac{2}{3}}\,(1-\xi)^{\frac{2}{3}}}$$

where the fractional complex powers are evaluated through a logarithm whose imaginary part lies between zero and π and the integration path can be chosen freely except for never entering the lower half plane. Although the denominator of the integrand can be written as

$$(1-\xi^2)^{\frac{2}{3}}$$

keeping the two factors apart emphasises the angle bending which the $-\frac{2}{3}$ powers are doing at the separate points -1 and 1 (which map to the two bottom corners of the triangle).



The above dimensions are easily calculated from the Schwarz-Christoffel mapping by direct integration. As the triangle is equilateral, the height is $\frac{\sqrt{3}}{2}$ times the base. Thus

$$\frac{\Gamma(\frac{1}{6}).\Gamma(\frac{1}{2})}{2.\Gamma(\frac{2}{3})} = \frac{\sqrt{3}}{2}\,\frac{\Gamma(\frac{1}{3})^2}{2^{\frac{1}{3}}\Gamma(\frac{2}{3})}$$

In other words

$$\boxed{\sqrt[3]{2}\,\Gamma(\tfrac{1}{6}).\Gamma(\tfrac{1}{2}) = \sqrt{3}\,\Gamma(\tfrac{1}{3})^2}$$

This is, therefore, a geometrical proof of a gamma function identity which we would otherwise have had to use both the reflection and duplication formulae to prove.

The picture shows the image of a grating in the upper half plane under this Schwarz-Christoffel mapping. We note that all grid lines meet at 90°. (They met at 90° in the upper half-plane and conformal mappings preserve angles.) The tear-drop pattern at the top is caused by the image of the uppermost grid line (parallel to the real axis but with a large imaginary value). It can be shown that the radius of curvature at the bottom of this tear-drop is asymptotically equal to one quarter of the remaining distance to the apex of the triangle. The vertical grid lines near the imaginary axis squeeze together proportionally to the fourth power of the remaining distance to the apex, making the central spike.

To generate the picture the complex integrations were carried out numerically. It should be noted that especial care is required close to the real axis because of the presence of the two cut-points at -1 and 1, and a finer numerical division was used there.

The derivation of the gamma function identity relies on the Schwarz-Christoffel mapping integrals from ξ=0 to ξ=1 and from ξ=0 to ξ=∞i both reducing to well-known standard real integrals.

Wikipedia have a good article on Schwarz-Christoffel mapping written in a similar vein, while *Driscoll and Trefethen*† treat much more complicated target domains including semi-infinite ones, generally mapping from the unit disc rather than the upper half-plane.

† Driscol TA, Trefethen LN (2002) "Schwarz-Christoffel mapping" Cambridge University Press.