The Riemann spaces related to the Navier-Stokes equations

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1. 6D-metrics and the NS-equations

For solving system of the Navier-Stokes equations

$$\frac{\partial}{\partial t}\vec{V}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}=\mu\Delta\vec{V}-\vec{\nabla}P(\vec{x},t),\quad \vec{\nabla}\cdot\vec{V}=0, \tag{1}$$

where $\vec{V}(\vec{x},t)$ -is the fluid velocity, $P(\vec{x},t)$- is the pressure and μ- is the viscosity of liquid, was used their representation in form of laws of conservations:

$$\frac{\partial}{\partial y}H(\vec{x},t)-\frac{\partial}{\partial x}E(\vec{x},t)=0,\quad \frac{\partial}{\partial z}H(\vec{x},t)-\frac{\partial}{\partial x}B(\vec{x},t)=0,$$

$$\frac{\partial}{\partial z}E(\vec{x},t)-\frac{\partial}{\partial y}B(\vec{x},t)=0, \tag{2}$$

where

$$H(\vec{x},t)=U_t+UU_x+VU_y+WU_z-\mu(U_{xx}+U_{yy}+U_{zz}),$$
$$E(\vec{x},t)=V_t+UV_x+VV_y+WV_z-\mu(V_{xx}+V_{yy}+V_{zz}),$$
$$B(\vec{x},t)=W_t+UW_x+VW_y+WW_z-\mu(W_{xx}+W_{yy}+W_{zz}).$$

This allow us to introduce six-dimension Riemann space, equipped with the metric

$$ds^2=-2\,B(\vec{x},t)dt\,dv+2\,E(\vec{x},t)dt\,dw+2\,H(\vec{x},t)dv\,dw-2\left(\int\frac{\partial}{\partial y}H(\vec{x},t)dz\right)dw^2+$$

$$+dt\,dx+dv\,dy+dw\,dz \tag{3}$$

and to use it for construct an examples of solutions of the NS-equations.

In general the metric (3) has fifteen components of the Ricci-tensor R_{ik}, and nine of them are equal to zero if the functions $H(\vec{x},t),E(\vec{x},t),B(\vec{x},t)$ satisfy the conditions (2). The remaining six components of the Ricci tensor $R_{vv},R_{vw},R_{ww},R_{tt},R_{tv},R_{tw}$ are expressed in terms of the functions $H(\vec{x},t),E(\vec{x},t),B(\vec{x},t)$ and their derivatives.

The Example.

The metric is the Ricci-flat- R_{ik}=0 at the conditions

$$B(\vec{x},t)=\frac{\partial}{\partial t}F(z,t)+F(z,t)\frac{\partial}{\partial z}F(z,t)-\mu\frac{\partial^2}{\partial z^2}F(z,t)$$

$$H(x,t)=-1/2\,x\frac{\partial^2}{\partial t\partial z}F(z,t)+1/4\,x\left(\frac{\partial}{\partial z}F(z,t)\right)^2-1/2\,xF(z,t)\frac{\partial^2}{\partial z^2}F(z,t)+1/2\,\mu\,x\frac{\partial^3}{\partial z^3}F(z,t)$$

$$E(x,t)=-1/2\,y\frac{\partial^2}{\partial t\partial z}F(z,t)+1/4\,y\left(\frac{\partial}{\partial z}F(z,t)\right)^2-1/2\,yF(z,t)\frac{\partial^2}{\partial z^2}F(z,t)+1/2\,\mu\,y\frac{\partial^3}{\partial z^3}F(z,t),$$

where

$$U(\vec{x},t)=-1/2\,x\frac{\partial}{\partial z}F(z,t),\quad V(\vec{x},t)=-1/2\,y\frac{\partial}{\partial z}F(z,t),\quad W(\vec{x},t)=F(z,t),$$

and

$$\frac{\partial^3}{\partial t\partial z^2}F(z,t)-F(z,t)\frac{\partial^3}{\partial z^3}F(z,t)+\mu\frac{\partial^4}{\partial z^4}F(z,t)=0.$$

2.

In particular case when the components of metric are of the form

$$B(\vec{x},t)=\frac{\partial^2}{\partial z^2}Q(\vec{x},t),\quad H(\vec{x},t)=\frac{\partial^2}{\partial x\partial z}Q(\vec{x},t)\quad E(\vec{x},t)=\frac{\partial^2}{\partial y\partial z}Q(\vec{x},t)$$

we get It takes form

$$ds^2=d_xd_t+d_yd_v+d_zd_w-2\left(\frac{\partial^2}{\partial z^2}Q(x,y,z,t)\right)d_td_v+2\left(\frac{\partial^2}{\partial y\partial z}Q(x,y,z,t)\right)d_td_w+$$

$$+2\left(\frac{\partial^2}{\partial x\partial z}Q(x,y,z,t)\right)d_vd_w-2\left(\frac{\partial^2}{\partial x\partial y}Q(x,y,z,t)\right){d_w}^2$$

and the solutions of the NS-equations are expressed throw the function $Q(\vec{x},t)=\frac{\partial P}{\partial y}$, that is solution of the Monge-Ampere equation (MA):

$$2\,Q_{xyzz}^2-2\,Q_{yyxz}Q_{xzzz}-2\,Q_{xxyz}Q_{yzzz}+Q_{xxzz}Q_{zzyy}+Q_{xxyy}Q_{zzzz}=0. \tag{4}$$

Theorem 1. ***The equation (1) determines function pressure $P(\vec{x},t)=\int(Q(\vec{x},t)dt)$ of flow and in particular cases admit reduction to the second order ODE of the form $y''+a_1(x,y)y'^3+3a_2(x,y)y'^2+3a_3(x,y)y'+a_4(x,y)=0$, where $a_i=a_i(x,y)$. This type of ODE's meet in theory of nonlinear dynamical systems $\dot{\vec{x}}=F_i(\vec{x})$ with polynomial right parts which have as the limit cycles and also strange attractors in their space of states.***

As an example, we give the equation

$$\frac{d^2}{dx^2}y(x)-3\,\frac{\left(\frac{d}{dx}y(x)\right)^2}{y(x)}-2\,\frac{(k-6\,x+3)\left(\frac{d}{dx}y(x)\right)y(x)}{k+3}-12\,\frac{(y(x))^3\,x^2}{k+2}+\frac{(4\,k+12)\,(y(x))^3\,x}{k+2}+$$

$$+\frac{(-k-2)\,(y(x))^3}{k+2}-6\,\frac{(y(x))^4\,x^4}{(k+3)\,(k+2)}-2\,\frac{(-6-2\,k)\,(y(x))^4\,x^3}{(k+3)\,(k+2)}-2\,\frac{\left(k^2+3\,k+3\right)(y(x))^4\,x^2}{(k+3)\,(k+2)}=0,$$

which has a form similar to the equation

$$y''-3\,\frac{y'^2}{y}+\left(\alpha\,y-x^{-1}\right)y'+\epsilon\,xy^4+\frac{\delta}{x}y^2-\gamma\,y^3-\beta\,x^3y^4-\beta\,x^2y^3=0,$$

which is equivalent to the Lorenz-dynamical system

$$\dot{y}=rx-y-xz,\ \ \dot{z}=xy-bz,\ \ \dot{x}=\sigma(y-x),$$

where:

$$\alpha=\frac{b+\sigma+1}{\sigma},\ \ \beta=\frac{1}{\sigma^2},\ \ \gamma=\frac{b(\sigma+1)}{\sigma^2},\ \ \delta=\frac{(\sigma+1)}{\sigma},\ \ \epsilon=\frac{b(r-1)}{\sigma^2}.$$

and solutions of the form

$$Q(x,y,z,t)=\_F1(x)+\_F2(y)+\_F3(z)+\_F0(t)+\_F1(x)\_F2(y)+\_F3(z)\_F2(y)+\_F1(x)\_F3(z)+\_F1(x)\_F3(z)\_F2(y),$$

where

$$\frac{d^2}{dx^2}\_F1(x)=\frac{\_c_1\left(\frac{d}{dx}\_F1(x)\right)^2}{1+\_F1(x)},$$

$$\frac{d^4}{dz^4}\_F3(z)=-\frac{\left(\frac{d^2}{dz^2}\_F3(z)\right)^2}{1+\_F3(z)}+2\,\frac{\left(\frac{d^3}{dz^3}\_F3(z)\right)\frac{d}{dz}\_F3(z)}{(1+\_F3(z))\,\_c_2}+\left(2\,\frac{\left(\frac{d^3}{dz^3}\_F3(z)\right)\frac{d}{dz}\_F3(z)}{1+\_F3(z)}-2\,\frac{\left(\frac{d^2}{dz^2}\_F3(z)\right)^2}{(1+\_F3(z))\,\_c_2}\right){\_c_1}^{-1},$$

$$\frac{d^2}{dy^2}\_F2(y)=\frac{\_c_2\left(\frac{d}{dy}\_F2(y)\right)^2}{1+\_F2(y)}.$$

where

$$z = \int \frac{\_j(\_h)}{\_h\, e^{\int \_j(\_h) d\_h + \_C6}} d\_h + \_C5,$$

$$\_F3(z) = e^{\int \frac{\_j(\_h)}{\_h} d\_h + \_C7} - 1$$

and the function $_j(_h) =$ **is solution of the first order ODE**

$$\frac{d}{d\_h}\_j(\_h) =$$

$$\frac{d}{d\_h}\_j(\_h) = 6\,\_h\,(\_j(\_h))^3 + 7\,(\_j(\_h))^2 + 2\,\frac{(-2\,\_c_1 + 1 + 6\,\_c_1\_c_2 - 2\,\_c_2)\,(\_j(\_h))^3}{\_c_1\_c_2} +$$

$$+\left(2\,\frac{(-3\,\_c_2 + 2 + 4\,\_c_1\_c_2 - 3\,\_c_1)\,(\_j(\_h))^3}{\_c_1\_c_2} + \_j(\_h) + \frac{(-2\,\_c_1 + 4\,\_c_1\_c_2 - 2\,\_c_2)\,(\_j(\_h))^2}{\_c_1\_c_2}\right)\_h^{-1} +$$

$$+2\,\frac{(-\_c_1 + \_c_1\_c_2 + 1 - \_c_2)\,(\_j(\_h))^3}{\_c_1\_c_2\_h^2}$$

with two parameters $\_c_2$ **and** $\_c_1$**.**

2. 14D-metrics and the NS-equations

A more general approach to study properties of the NS**-equations is connected by using the** 14**-dimension space with local coordinates** $x, y, z, t, u, v, w, p, \xi, \eta, \chi, \rho, q, \delta$**.**

Theorem 2. The metric

$$ds^2 = 2\,dxdu + 2\,dydv + 2\,dzdw + (-W(\vec{x},t)w - V(\vec{x},t)v - U(\vec{x},t)u)\,dt^2 +$$

$$+\left(-U(\vec{x},t)p - u\,(U(\vec{x},t))^2 - uP(\vec{x},t) + w\mu\,\frac{\partial}{\partial z}U(\vec{x},t) - wU(\vec{x},t)W(\vec{x},t)\right)d\eta^2 +$$

$$+\left(v\mu\,\frac{\partial}{\partial y}U(\vec{x},t) - vU(\vec{x},t)V(\vec{x},t) + u\mu\,\frac{\partial}{\partial x}U(\vec{x},t)\right)d\eta^2 + 2\,d\eta d\xi + 2\,d\rho d\chi + 2\,dmdn +$$

$$\left(-V(\vec{x},t)p - vP(\vec{x},t) - v\,(\vec{x},t))^2 - V(\vec{x},t)W(\vec{x},t)w + v\mu\,\frac{\partial}{\partial y}V(\vec{x},t) - uU(\vec{x},t)V(\vec{x},t)\right)d\rho^2 +$$

$$\left(u\mu\,\frac{\partial}{\partial x}V(\vec{x},t)\right)d\rho^2 + \left(-uU(\vec{x},t)W(\vec{x},t) - w\,(W(\vec{x},t))^2 - wP(\vec{x},t) + w\mu\,\frac{\partial}{\partial z}W(\vec{x},t)\right)dm^2 +$$

$$\left(v\mu\,\frac{\partial}{\partial y}W(\vec{x},t) - vV(\vec{x},t)W(\vec{x},t) + u\mu\,\frac{\partial}{\partial x}W(\vec{x},t) - W(\vec{x},t)p\right)dm^2 \qquad (1)$$

with the components depending from the functions U, V, W, P **has four components of the Ricci-tensor which are zero**

$$R_{44} = U_x + V_y + W_z = 0,\;\; R_{55} = 0,\;\; R_{66} = 0,\;\; R_{77} = 0,$$

on solutions of the system (1), i.e. the Ricci-flat $R_{ik} = 0$ **on solutions of the NS-equations.**

To deriving this metric are a used law of conservations of NS-system of the form

$$U_t + (U^2 - \mu U_x + P)_x + (UV - \mu U_y)_y + (UW - \mu U_z)_z = 0$$

$$V_t + (V^2 - \mu V_y + P)_y + (UV - \mu V_x)_x + (VW - \mu V_z)_z = 0,$$

$$W_t + (W^2 - \mu W_z + P)_z + (UW - \mu W_x)_x + (VW - \mu W_y)_y = 0,$$

$$U_x + V_y + W_z) = 0.$$

The metric is the metric of Riemann extensions

$$^{14}ds^2 = -2\Gamma^i{}_{jk}(\vec{x},t)\Psi_i dx^j dx^k + 2d\Psi_l dx^l$$

of $7D$**-affine connection in local ooordinates** $x, y, z, t, \eta, \rho, m$**, defined by conditions**

$$R_{ij} = \partial_k \Gamma^i jk - \partial_i \Gamma^k jk + \Gamma^k lk \Gamma^l ji - \Gamma^k im \Gamma^m jk.$$

The metric (1) belongs to the class of the Riemann spaces with vanishing scalar Invariants (meet in theory of gravitation waves!!!) and part of its geodesics with respect to the coordinates $\eta, \rho, m, \xi, \chi, n$ **are determined by the equations**

$$\ddot{\eta} = 0,\ \ddot{\rho} = 0,\ \ddot{m} = 0,\ \ddot{\xi} = 0,\ \ddot{\chi} = 0,\ \ddot{n} = 0.$$

and have the form

$$\eta(s) = a_1\, s + b_1, \quad \rho(s) = a_2\, s + b_2, \quad m(s) = a_3\, s + b_3, \quad \xi(s) = a_4\, s + b_4,$$

$$\chi(s) = a_5\, s + b_5, \quad n(s) = a_6\, s + b_6.$$

The equations to the coordinates $[x, y, z, t]$ **take the form**

$$\frac{d^2}{ds^2}x(s) - 1/2\,U\left(\frac{d}{ds}t(s)\right)^2 + 1/2\,{a_1}^2\mu\,\frac{\partial}{\partial x}U - 1/2\,{a_1}^2\,(U))^2 - 1/2\,a1^2P - 1/2\,{a_2}^2UV + 1/2\,a2^2\mu\,\frac{\partial}{\partial x}V -$$

$$-1/2\,{a_3}^2UW + 1/2\,{a_3}^2\mu\,\frac{\partial}{\partial x}W = 0,$$

$$\frac{d^2}{ds^2}y(s) - 1/2\,V\left(\frac{d}{ds}t(s)\right)^2 - 1/2\,{a_1}^2UV + 1/2\,{a_1}^2\mu\,\frac{\partial}{\partial y}U + 1/2\,{a_2}^2\mu\,\frac{\partial}{\partial y}V - 1/2\,{a_2}^2\,(V)^2 - 1/2\,{a_2}^2P -$$

$$-1/2\,{a_3}^2VW + 1/2\,{a_3}^2\mu\,\frac{\partial}{\partial y}W = 0,$$

$$\frac{d^2}{ds^2}z(s) - 1/2\,W\left(\frac{d}{ds}t(s)\right)^2 - 1/2\,{a_1}^2UW + 1/2\,{a_1}^2\mu\,\frac{\partial}{\partial z}U - 1/2\,{a_2}^2VW + 1/2\,{a_2}^2\mu\,\frac{\partial}{\partial z}V +$$

$$+1/2\,{a_3}^2\mu\,\frac{\partial}{\partial z}W - 1/2\,{a_3}^2\,(W)^2 - 1/2\,{a_3}^2P = 0,$$

$$\frac{d^2}{ds^2}t(s) - 1/2\,U{a_1}^2 - 1/2\,V{a_2}^2 - 1/2\,W{a_3}^2 = 0,$$

and the coordinates $[u, v, w, p]$ **satisfy the linear system of ODE with respect to the** $u^i = [u, v, w, p]$ **with the coefficients depended on the** (x, y, z, t) **and looks as**

$$\frac{d^2}{ds^2}u(s) = A_1\,u(s) + B_1\,v(s) + C_1\,w(s) + E_1\,p(s),$$

where

$$A_1 = 1/2\,{a_3}^2\left(\frac{\partial}{\partial x}U\right)W - 1/2\,{a_1}^2\mu\,\frac{\partial^2}{\partial x^2}U + {a_1}^2U\frac{\partial}{\partial x}U - 1/2\,{a_2}^2\mu\,\frac{\partial^2}{\partial x^2}V + 1/2\,{a_3}^2U\frac{\partial}{\partial x}W -$$

$$-1/2\,{a_3}^2\mu\,\frac{\partial^2}{\partial x^2}W + 1/2\,{a_2}^2\left(\frac{\partial}{\partial x}U\right)V + 1/2\left(\frac{d}{ds}t(s)\right)^2\frac{\partial}{\partial x}U + 1/2\,{a_2}^2U\frac{\partial}{\partial x}V + 1/2\,{a_1}^2\frac{\partial}{\partial x}P,$$

$$B_1 = {a_2}^2V\frac{\partial}{\partial x}V - 1/2\,{a_1}^2\mu\,\frac{\partial^2}{\partial x\partial y}U - 1/2\,{a_2}^2\mu\,\frac{\partial^2}{\partial x\partial y}V + 1/2\,{a_2}^2\frac{\partial}{\partial x}P + 1/2\,{a_3}^2\left(\frac{\partial}{\partial x}V\right)W +$$

$$+1/2\,{a_3}^2V\frac{\partial}{\partial x}W - 1/2\,{a_3}^2\mu\,\frac{\partial^2}{\partial x\partial y}W + 1/2\left(\frac{d}{ds}t(s)\right)^2\frac{\partial}{\partial x}V + 1/2\,{a_1}^2\left(\frac{\partial}{\partial x}U\right)V + 1/2\,{a_1}^2U\frac{\partial}{\partial x}V,$$

$$C_1 = {a_3}^2W\frac{\partial}{\partial x}W + 1/2\,{a_1}^2U\frac{\partial}{\partial x}W - 1/2\,{a_1}^2\mu\,\frac{\partial^2}{\partial x\partial z}U + 1/2\,{a_1}^2\left(\frac{\partial}{\partial x}U\right)W - 1/2\,a3^2\mu\,\frac{\partial^2}{\partial x\partial z}W +$$

$$+1/2\, {a_3}^2 \frac{\partial}{\partial x} P + 1/2 \left(\frac{d}{ds} t(s)\right)^2 \frac{\partial}{\partial x} W + 1/2\, {a_2}^2 \left(\frac{\partial}{\partial x} V\right) W + 1/2\, a2^2 V \frac{\partial}{\partial x} W - 1/2\, {a_2}^2 \mu \frac{\partial^2}{\partial x \partial z} V,$$

$$E_1 = 1/2\, {a_2}^2 \frac{\partial}{\partial x} V + 1/2\, {a_1}^2 \frac{\partial}{\partial x} U + 1/2\, {a_3}^2 \frac{\partial}{\partial x} W,$$

and

$$\frac{d^2}{ds^2} v(s) = A_2\, u(s) + B_2\, v(s) + C_2\, w(s) + E_2\, p(s),$$

$$\frac{d^2}{ds^2} w(s) = A_3\, u(s) + B_3\, v(s) + C_3\, w(s) + E_3\, p(s),$$

$$\frac{d^2}{ds^2} p(s) = A_4\, u(s) + B_4\, v(s) + C_4\, w(s) + E_4\, p(s),$$

where

$$E_4 = 1/2\, {a_3}^2 \frac{\partial}{\partial t} W + 1/2\, {a_2}^2 \frac{\partial}{\partial t} V + 1/2\, {a_1}^2 \frac{\partial}{\partial t} U,$$

$$C4 = 1/2\, {a_3}^2 (W)^2 + 1/2\, {a_1}^2 \left(\frac{\partial}{\partial t} U\right) W + 1/2\, {a_2}^2 V W + 1/2\, {a_3}^2 \frac{\partial}{\partial t} P + 1/2 \left(\frac{d}{ds} t(s)\right)^2 \frac{\partial}{\partial t} W +$$

$$+ {a_3}^2 W \frac{\partial}{\partial t} W + 1/2\, {a_1}^2 U W + 1/2\, {a_1}^2 U \frac{\partial}{\partial t} W - 1/2\, {a_1}^2 \mu \frac{\partial^2}{\partial t \partial z} U - 1/2\, {a_3}^2 \mu \frac{\partial^2}{\partial t \partial z} W -$$

$$-1/2\, {a_2}^2 \mu \frac{\partial^2}{\partial t \partial z} V + 1/2\, {a_2}^2 V \frac{\partial}{\partial t} W + 1/2\, {a_2}^2 \left(\frac{\partial}{\partial t} V\right) W,$$

$$B4 = 1/2\, {a_1}^2 \left(\frac{\partial}{\partial t} U\right) V + 1/2\, {a_2}^2 \frac{\partial}{\partial t} P + 1/2\, {a_2}^2 (V)^2 - 1/2\, {a_2}^2 \mu \frac{\partial^2}{\partial t \partial y} V + 1/2\, {a_1}^2 U \frac{\partial}{\partial t} V +$$

$$+1/2 \left(\frac{d}{ds} t(s)\right)^2 \frac{\partial}{\partial t} V + 1/2\, {a_3}^2 \left(\frac{\partial}{\partial t} V\right) W + 1/2\, {a_3}^2 V \frac{\partial}{\partial t} W + 1/2\, {a_1}^2 U V - 1/2\, {a_1}^2 \mu \frac{\partial^2}{\partial t \partial y} U +$$

$$+1/2\, {a_3}^2 V W + {a_2}^2 V \frac{\partial}{\partial t} V - 1/2\, {a_3}^2 \mu \frac{\partial^2}{\partial t \partial y} W,$$

$$A_4 = {a_1}^2 U \frac{\partial}{\partial t} U + 1/2\, {a_1}^2 \frac{\partial}{\partial t} P - 1/2\, {a_2}^2 \mu \frac{\partial^2}{\partial t \partial x} V + 1/2\, {a_1}^2 (U)^2 + 1/2\, {a_3}^2 U \frac{\partial}{\partial t} W - 1/2\, {a_3}^2 \mu \frac{\partial^2}{\partial t \partial x} W +$$

$$+1/2\, {a_2}^2 U V + 1/2\, {a_3}^2 \left(\frac{\partial}{\partial t} U\right) W + 1/2\, {a_2}^2 \left(\frac{\partial}{\partial t} U\right) V + 1/2\, {a_2}^2 U \frac{\partial}{\partial t} V + 1/2\, {a_3}^2 U W +$$

$$+1/2 \left(\frac{d}{ds} t(s)\right)^2 \frac{\partial}{\partial t} U - 1/2\, {a_1}^2 \mu \frac{\partial^2}{\partial t \partial x} U.$$

1 On the metrics with vanishing scalar Invariants

For the metric (1), all scalar invariants constructed from the Riemann tensor and its covariant derivatives are equal to zero. This circumstance forces us to use invariants of another type - the Cartan invariants

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K:=R{^i a j b}*R{^j c i d}* m^{a}*m^{b}*m^{c}*m^{d}
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S:=R{a b}*R{c d} n{^a}*n{^b}*n{c}*n{^d}
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As example

$$K = (U(x,y,z,t))^2 \frac{\partial}{\partial x} U(x,y,z,t) + \left(\frac{\partial}{\partial z} W(x,y,z,t)\right) (W(x,y,z,t))^2 + (U(x,y,z,t))^2 \frac{\partial}{\partial z} W(x,y,z,t) +$$

$$+ \left(\frac{\partial}{\partial y} V(x,y,z,t)\right) (W(x,y,z,t))^2 + (U(x,y,z,t))^2 \frac{\partial}{\partial y} V(x,y,z,t) + \left(\frac{\partial}{\partial x} U(x,y,z,t)\right) (W(x,y,z,t))^2 -$$

$$-2\left(\frac{\partial}{\partial z}W(x,y,z,t)\right)U(x,y,z,t)W(x,y,z,t)-2\left(\frac{\partial}{\partial x}U(x,y,z,t)\right)U(x,y,z,t)W(x,y,z,t)-$$
$$-2\left(\frac{\partial}{\partial y}V(x,y,z,t)\right)U(x,y,z,t)W(x,y,z,t)$$

Another way to study the properties of the NS system is to use the differential parameters of Beltrami

If $f(x^i)$ and $h(x^i)$ are the functions of coordinates of the space then the functions defined as

$$\Delta_1(f)=g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j},\quad \Delta_1(f,h)=g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_j},$$
$$\Delta_2(f)=g^{ij}\frac{\partial^2 f}{\partial x_i,x_j}-\Gamma^k_{ij}\frac{\partial f}{\partial x_k}$$

In the case

$$f=\psi(x,y,z,t,\eta,\rho,m,u,v,w,p,\xi,\chi,n)$$

from the Laplace-Beltrami equation

$$\Delta(\psi)=0$$

in particular we find the equation

$$-\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)uP(x,y,z,t)-\left(\frac{\partial^2}{\partial p^2}\_F15\right)W(x,y,z,t)w-\left(\frac{\partial^2}{\partial p^2}\_F15\right)U(x,y,z,t)u-$$
$$-\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)V(x,y,z,t)p-\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)U(x,y,z,t)p-\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)v\left(V(x,y,z,t)\right)^2-$$
$$-\left(\frac{\partial^2}{\partial p^2}\_F15\right)V(x,y,z,t)v-\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)u\left(U(x,y,z,t)\right)^2-\left(\frac{\partial^2}{\partial n^2}\_F15\right)w\left(W(x,y,z,t)\right)^2-$$
$$-\left(\frac{\partial^2}{\partial n^2}\_F15\right)wP(x,y,z,t)-\left(\frac{\partial^2}{\partial n^2}\_F15\right)W(x,y,z,t)p-\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)vP(x,y,z,t)+$$
$$+\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)w\mu\frac{\partial}{\partial z}V(x,y,z,t)+\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)v\mu\frac{\partial}{\partial y}U(x,y,z,t)-\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)vU(x,y,z,t)V(x,y,z,t)+$$
$$+\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)u\mu\frac{\partial}{\partial x}V(x,y,z,t)+\left(\frac{\partial^2}{\partial n^2}\_F15\right)w\mu\frac{\partial}{\partial z}W(x,y,z,t)+\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)u\mu\frac{\partial}{\partial x}U(x,y,z,t)-$$
$$-\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)uU(x,y,z,t)V(x,y,z,t)-\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)wU(x,y,z,t)W(x,y,z,t)+$$
$$+\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)v\mu\frac{\partial}{\partial y}V(x,y,z,t)-\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)wV(x,y,z,t)W(x,y,z,t)-$$
$$-\left(\frac{\partial^2}{\partial n^2}\_F15\right)uU(x,y,z,t)W(x,y,z,t)+\left(\frac{\partial^2}{\partial n^2}\_F15\right)u\mu\frac{\partial}{\partial x}W(x,y,z,t)-$$
$$-\left(\frac{\partial^2}{\partial n^2}\_F15\right)vV(x,y,z,t)W(x,y,z,t)+\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)w\mu\frac{\partial}{\partial z}U(x,y,z,t)+\left(\frac{\partial^2}{\partial n^2}\_F15\right)v\mu\frac{\partial}{\partial y}W(x,y,z,t)-$$
$$-2\left(\frac{\partial}{\partial n}\_F15\right)\_c_7-2\left(\frac{\partial}{\partial \xi}\_F15\right)\_c_5-2\left(\frac{\partial}{\partial \chi}\_F15\right)\_c_6-2\frac{\partial^2}{\partial v\partial y}\_F15-$$
$$-2\frac{\partial^2}{\partial p\partial t}\_F15-2\frac{\partial^2}{\partial u\partial x}\_F15-2\frac{\partial^2}{\partial w\partial z}\_F15=0,$$

where $\_F15=\_F15((x,y,z,t,u,v,w,p,\xi,\chi,n))$ **and the relation of the form**

$$(U(x,y,z,t)-W(x,y,z,t))P(x,y,z,t)/\mu=\left(\frac{\partial}{\partial z}U(x,y,z,t)\right)U(x,y,z,t)-W(x,y,z,t)\frac{\partial}{\partial x}V(x,y,z,t)-$$

$$-W(x,y,z,t)\frac{\partial}{\partial x}U(x,y,z,t) - W(x,y,z,t)\frac{\partial}{\partial x}W(x,y,z,t) + \left(\frac{\partial}{\partial z}V(x,y,z,t)\right)U(x,y,z,t)+$$

$$+\left(\frac{\partial}{\partial z}W(x,y,z,t)\right)U(x,y,z,t), \qquad (2)$$

when

$$\psi(x,y,z,t,\eta,\rho,m,u,v,w,p,\xi,\chi,n) = A(x,y,z,t,p)e^{-\eta-\xi-m-n-\chi-\rho}.$$

With the help of the condition (2) we find as particular case the system of equations 1.

$$\left(\frac{\partial^2}{\partial z^2}\phi(x,y,z,t)\right)\frac{\partial^2}{\partial x^2}\phi(x,y,z,t) - \left(\frac{\partial^2}{\partial x\partial z}\phi(x,y,z,t)\right)^2 + F(x,z,t=0,$$

$$\frac{\partial^3}{\partial x\partial t\partial x}\phi(x,y,z,t)+\frac{\partial^3}{\partial z\partial t\partial z}\phi(x,y,z,t)+\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\left(\frac{\partial^3}{\partial x^3}\phi(x,y,z,t) + \frac{\partial^3}{\partial z\partial x\partial z}\phi(x,y,z,t)\right)-$$

$$-\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial^3}{\partial x^2\partial z}\phi(x,y,z,t) + \frac{\partial^3}{\partial z^3}\phi(x,y,z,t)\right)-$$

$$-\mu\left(\frac{\partial^4}{\partial x^4}\phi(x,y,z,t) + 2\,\frac{\partial^4}{\partial z\partial x^2\partial z}\phi(x,y,z,t) + \frac{\partial^4}{\partial z^4}\phi(x,y,z,t)\right)-$$

$$-\mu\left(\frac{\partial^4}{\partial y\partial x^2\partial y}\phi(x,y,z,t) + \frac{\partial^4}{\partial z\partial y^2\partial z}\phi(x,y,z,t)\right) = 0,$$

which give as the solution

$$U(x,y,z,t) = \frac{y - x + xt - tK(z,t)\sin(t^{-1})}{t^2},$$

$$V(x,y,z,t) = \frac{y - 2\,x + yt - tK(z,t)\left(\sin(t^{-1}) + \cos(t^{-1})\right)}{t^2},$$

$$W(x,y,z,t) = -2\,\frac{z}{t},$$

depending from solutions of the Heat equation

$$\frac{\partial}{\partial t}K(z,t) + \mu\,\frac{\partial^2}{\partial z^2}K(z,t) + 2\,\frac{z\frac{\partial}{\partial z}K(z,t)}{t}.$$

This metric belongs to the class of partially-projective spaces and their Cartan-Invariants,

$$C = R(^i\ a\ j\ b)R(^j\ c\ i\ d)A^aA^bA^cA^d,$$

where A^i 14D-vector constructed on the basis of the Riemann curvature $C = R(^iaj\ b)$ of the space can be used in theory of the NS-equations.

2 On the limit cycles and strange attractors

From the Euler system $\mu = 0$

$$\frac{\partial}{\partial t}\vec{V} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V} + \vec{\nabla}P(\vec{x}) = 0, \quad \vec{\nabla}\cdot\vec{V} = 0,$$

after substitution

$$U(\vec{x},t) = A_1(t)x + A_2(t)y + A_3(t)z,\ \ V(\vec{x},t) = B_1(t)x + B_2(t)y + B_3(t)z,$$

$$W(\vec{x},t) = C_1(t)x + C_2(t)y + C_3(t)z, \quad B_2(t) = -A_1(t) - C3(t)$$

we get the conditions on the coefficients

$$\frac{d}{dt}A_2(t) = -B_1\,C_3 + A_2\,C_3 - A_3\,C_2 + \frac{d}{dt}B_1(t) + B_3\,C_1,$$

$$\frac{d}{dt}B_3(t) = C_1\,A_2 - B_1\,A_3 + B_3\,A_1 + \frac{d}{dt}C_2(t) - C_2\,A_1,$$

$$\frac{d}{dt}C_1(t) = A_1\,A_3 + A_2\,B_3 + \frac{d}{dt}A_3(t) + C_3\,A_3 - C_1\,A_1 - C_2\,B_1 - C_3\,C_1. \qquad (3)$$

In special case these relations take the form

$$\frac{d}{dt}A_2(t) = 4\,a_0\,{C_1}^2 + (4\,a_2\,B_3 + (3\,a_1 - b_2)\,A_2)\,C_1 + (2\,a11 - b_1 2)\,{A_2}^2 + (3\,a12 - 2\,b22)\,A_2\,B_3 + 4\,a22\,{B_3}^2,$$

$$\frac{d}{dt}B_3(t) = 4\,b\_0\,{C_1}^2 + (4\,b\_1\,A_2 + (3\,b_2 - a_1)\,B_3)\,C_1 + (2\,b22 - a12)\,{B_3}^2 + (3\,b12 - 2\,a11)\,A_2\,B_3 + 4\,b11\,{A_2}^2,$$

$$\frac{d}{dt}C_1(t) = (-a_1 - b_2)\,{C_1}^2 + (-2\,b22 - a12)\,C_1\,B_3 + (-b12 - 2\,a11)\,A_2\,C_1 \qquad (4)$$

The system (4) is the projective extension of the polynomial system

$$\frac{d}{dt}x(t) = a0 + a1\,x + a2\,y + a11\,x^2 + a12\,xy + a22\,y^2,$$

$$\frac{d}{dt}y(t) = b0 + b1\,x + b2\,y + b11\,x^2 + b12\,xy + b22\,y^2.$$

It has the limit cycles at the special choice of parameters $a_i, b_i, a_(ij), b_(ij)$ (the numbers of the limit cycles in this system- is unsolved up to this time Hilbert-problem!!!)

Remark that the reductions at the R
ossler system with limit cycles

$$\frac{d}{dt}A_2(t) = -B_3(t) - C_1(t), \quad \frac{d}{dt}B_3(t) = \alpha\,B_3(t) + A_2(t),$$

$$\frac{d}{dt}C_1(t) = C_1(t)\,(A_2(t) - \delta) + \beta,$$

and at the Lorenz system

$$\frac{d}{dt}A_2(t) = \sigma\,(B_3(t) - A_2(t))\,, \quad \frac{d}{dt}B_3(t) = -B_3(t) - A_2(t)C_1(t) + rA_2(t),$$

$$\frac{d}{dt}C_1(t) = -bC_1(t) + A_2(t)B_3(t)$$

The case of full the NS-system $\mu \neq 0$ is a more complicate.

As example in the case when the components of the velocities are of the form (V.V. Pukhnachov, 1972)

$$V(x,y,z,t) = \frac{\partial}{\partial x}H(x,y,z,t), \quad W(x,y,z,t) = W1(y,t), \quad P(x,y,z,t) = P1(y,t),$$

$$U(x,y,z,t) = -\frac{\partial}{\partial y}H(x,y,z,t),$$

properties of the flows depend from the solutions of the function $H(x,y,z,t)$ satisfying to the PDE

$$-\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2 \frac{\partial^3}{\partial t^2\partial y}H(\vec{x},t) - 2\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)^2 \mu^2 \frac{\partial^3}{\partial z\partial y\partial z}H(\vec{x},t) -$$

$$-2\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)^2 \mu^2 \frac{\partial^3}{\partial x^2\partial y}H(\vec{x},t) + \left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2 \mu\,\frac{\partial^4}{\partial x^2\partial t\partial y}H(\vec{x},t) -$$

$$-\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)^2 \left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2 \frac{\partial^3}{\partial y^3}H(\vec{x},t) + \left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2 \left(\frac{\partial^2}{\partial t\partial y}H(\vec{x},t)\right) \frac{\partial^2}{\partial x\partial y}H(\vec{x},t) -$$

$$-2\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)^2 \mu^2 \frac{\partial^3}{\partial y^3}H(\vec{x},t) + \left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\left(\frac{\partial^4}{\partial y^3\partial z}H(\vec{x},t)\right)\mu^2 \frac{\partial^3}{\partial y^3}H(\vec{x},t) -$$

$$-\left(\frac{\partial^3}{\partial y\partial t\partial z}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\left(\frac{\partial}{\partial y}H(\vec{x},t)\right)\frac{\partial^2}{\partial x\partial y}H(\vec{x},t)+$$

$$+\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\mu\,\frac{\partial^4}{\partial z\partial y^2\partial z}H(\vec{x},t)+\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\mu\,\frac{\partial^4}{\partial y\partial x^2\partial y}H(\vec{x},t)-$$

$$-\left(\frac{\partial^3}{\partial y\partial t\partial z}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\mu\,\frac{\partial^3}{\partial z\partial y\partial z}H(\vec{x},t)-\left(\frac{\partial^3}{\partial y\partial t\partial z}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\mu\,\frac{\partial^3}{\partial y^3}H(\vec{x},t)-$$

$$-\left(\frac{\partial^3}{\partial y\partial t\partial z}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\mu\,\frac{\partial^3}{\partial x^2\partial y}H(\vec{x},t)+\left(\frac{\partial^3}{\partial y\partial t\partial z}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\frac{\partial^2}{\partial y^2}H(\vec{x},t)-$$

$$-\mu\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\left(\frac{\partial^4}{\partial y^3\partial z}H(\vec{x},t)\right)\frac{\partial^2}{\partial t\partial y}H(\vec{x},t)-\mu\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\left(\frac{\partial}{\partial y}H(\vec{x},t)\right)\frac{\partial^4}{\partial y^2\partial x\partial y}H(\vec{x},t)-$$

$$-\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)\mu\,\frac{\partial^3}{\partial z\partial y\partial z}H(\vec{x},t)-$$

$$-\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(x,y,z,t)\right)\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)\mu\,\frac{\partial^3}{\partial x^2\partial y}H(\vec{x},t)-$$

$$-\mu\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\left(\frac{\partial^4}{\partial y^3\partial z}H(\vec{x},t)\right)\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\frac{\partial^2}{\partial y^2}H(\vec{x},t)+$$

$$+\mu\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\left(\frac{\partial^4}{\partial y^3\partial z}H(\vec{x},t)\right)\left(\frac{\partial}{\partial y}H(\vec{x},t)\right)\frac{\partial^2}{\partial x\partial y}H(\vec{x},t)-$$

$$-\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t\vec{x},t)\right)\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)\left(\frac{\partial}{\partial y}H(\vec{x},t)\right)\frac{\partial^2}{\partial x\partial y}H(\vec{x},t)-$$

$$-\mu\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\left(\frac{\partial^2}{\partial y^2}H(\vec{x},t)\right)\frac{\partial^3}{\partial y\partial x\partial y}H(\vec{x},t)+2\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\mu^2\frac{\partial^4}{\partial y^4}H(\vec{x},t)-$$

$$-2\,\mu\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)^2\left(\frac{\partial}{\partial y}H(\vec{x},t)\right)\frac{\partial^2}{\partial x\partial y}H(x,y,z,t)+2\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\mu\,\frac{\partial^4}{\partial y^4}H(\vec{x},t)+$$

$$+2\,\mu\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)^2\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\frac{\partial^2}{\partial y^2}H(\vec{x},t)-2\,\mu\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\frac{\partial^3}{\partial y\partial t\partial y}H(\vec{x},t)+$$

$$+2\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\mu^2\frac{\partial^4}{\partial z\partial y^2\partial z}H(\vec{x},t)+$$

$$+\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\left(\frac{\partial}{\partial y}H(\vec{x},t)\right)\frac{\partial^3}{\partial y\partial x\partial y}H(\vec{x},t)+$$

$$+\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\left(\frac{\partial^4}{\partial y^3\partial z}H(\vec{x},t)\right)\mu^2\frac{\partial^3}{\partial z\partial y\partial z}H(\vec{x},t)+$$

$$+\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(x,y,z,t)\right)\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)\frac{\partial^2}{\partial t\partial y}H(\vec{x},t)+$$

$$+\mu\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\left(\frac{\partial^2}{\partial x\partial y}H(\vec{x},t)\right)\frac{\partial^3}{\partial y^3}H(\vec{x},t)+$$

$$+\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)^2\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)\frac{\partial^2}{\partial y^2}H(\vec{x},t)+$$

$$+\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\left(\frac{\partial^4}{\partial y^3\partial z}H(\vec{x},tt)\right)\mu^2\frac{\partial^3}{\partial x^2\partial y}H(\vec{x},t)+$$

$$+2\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\mu^2\frac{\partial^4}{\partial y\partial x^2\partial y}H(\vec{x},t)-$$

$$-3\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)\mu\,\frac{\partial^3}{\partial y^3}H(\vec{x},t)+$$

$$+2\,\mu\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\left(\frac{\partial}{\partial y}H(\vec{x},t)\right)\frac{\partial^3}{\partial y\partial x\partial y}H(\vec{x},t)+2\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\mu\,\frac{\partial^4}{\partial y^2\partial t\partial y}H(\vec{x},t)-$$

$$-2\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\left(\frac{\partial}{\partial x}H(\vec{x},t)\right)\frac{\partial^3}{\partial y\partial t\partial y}H(\vec{x},t)-\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\left(\frac{\partial^2}{\partial t\partial x}H(\vec{x},t)\right)\frac{\partial^2}{\partial y^2}H(x,y,z,t)+$$

$$+\left(\frac{\partial^3}{\partial y\partial t\partial z}H(\vec{x},t)\right)\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)\frac{\partial^2}{\partial t\partial y}H(\vec{x},t)+\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\mu\,\frac{\partial^4}{\partial z\partial y\partial t\partial z}H(\vec{x},t)+$$

$$+\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\left(\frac{\partial}{\partial y}H(\vec{x},t)\right)\frac{\partial^3}{\partial x\partial t\partial y}H(\vec{x},t)-$$

$$-\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\mu^2\frac{\partial^5}{\partial z\partial y^3\partial z}H(\vec{x},t)-\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\mu^2\frac{\partial^5}{\partial y^2\partial x^2\partial y}H(\vec{x},t)-$$

$$-\left(\frac{\partial^2}{\partial y\partial z}H(\vec{x},t)\right)^2\mu^2\frac{\partial^5}{\partial y^5}H(\vec{x},t)+2\,\mu\left(\frac{\partial^3}{\partial y^2\partial z}H(\vec{x},t)\right)^2\frac{\partial^2}{\partial t\partial y}H(\vec{x},t)=0.$$

This equation admit reduction at the ODE of the form

$$\left(\left(D^{(2)}\right)(H1)(\xi)\right)^2\mu\left(D^{(5)}\right)(H1)(\xi)-\mathbf{D}(H1)(\xi)\left(\left(D^{(2)}\right)(H1)(\xi)\right)^2\left(D^{(4)}\right)(H1)(\xi)-$$

$$-3\left(D^{(2)}\right)(H1)(\xi)\left(D^{(4)}\right)(H1)(\xi)\mu\left(D^{(3)}\right)(H1)(\xi)+$$

$$+\left(\left(D^{(3)}\right)(H1)(\xi)\right)^2\mathbf{D}(H1)(\xi)\left(D^{(2)}\right)(H1)(\xi)+2\left(\left(D^{(3)}\right)(H1)(\xi)\right)^3\mu+$$

$$+\left(\left(D^{(2)}\right)(H1)(\xi)\right)^2\left(D^{(4)}\right)(H1)(\xi)k-\left(\left(D^{(3)}\right)(H1)(\xi)\right)^2\left(D^{(2)}\right)(H1)(\xi)k=0,$$

where $\xi=x+y+z-kt$**.**

Solution of this equation is expressed throw the second order ODE

$$=0 \tag{5}$$

3 Lorenz system

The system

$$\dot{y}=rx-y-xz,\ \ \dot{z}=xy-bz,\ \ \dot{x}=\sigma(y-x),$$

is reduced to the second order ODE

$$y''-3\,\frac{y'^2}{y}+\left(\alpha\,y-x^{-1}\right)y'+\epsilon\,xy^4+\frac{\delta}{x}y^2-\gamma\,y^3-\beta\,x^3y^4-\beta\,x^2y^3=0,$$

where

$$\alpha=\frac{b+\sigma+1}{\sigma},\ \ \beta=\frac{1}{\sigma^2},\ \ \gamma=\frac{b(\sigma+1)}{\sigma^2},\ \ \delta=\frac{(\sigma+1)}{\sigma},\ \ \epsilon=\frac{b(r-1)}{\sigma^2}.$$

The equation (5) after change of variables looks as

$$\frac{d^2}{dx^2}y(x)-3\,\frac{\left(\frac{d}{dx}y(x)\right)^2}{y}+\left(\left(5-\frac{1}{x\mu}\right)y-x^{-1}\right)\frac{d}{dx}y(x)-\left(-4\,x+2\,\mu^{-1}\right)y^4+\left(10-3\,\frac{1}{x\mu}\right)y^3+3\,\frac{y^2}{x}=0.$$

This equation has the invariant form

$$\frac{d^2}{dx^2}y(x)+a1(x,y)\left(\frac{d}{dx}y(x)\right)^3+3\,a2(x,y)\left(\frac{d}{dx}y(x)\right)^2+3\,a3(x,y)\frac{d}{dx}y(x)+a4(x,y)=0$$

under change of the variables $x = x(u, v),\ \ y = y(u, v)$ **and has the invariants**

$$Nu\_(m+5) = L\_1\,\frac{\partial}{\partial y}Nu\_m - L\_2\,\frac{\partial}{\partial x}Nu\_m + mNu\_m\left(\frac{\partial}{\partial x}L\_2(x,y) - \frac{\partial}{\partial y}L\_1(x,y)\right),$$

constructed from

$$Nu\_5 = L\_2\left(L\_1\,\frac{\partial}{\partial x}L\_2(x,y) - L\_2\,\frac{\partial}{\partial x}L\_1(x,y)\right) + L\_1\left(L\_2\,\frac{\partial}{\partial y}L\_1(x,y) - L\_1\,\frac{\partial}{\partial y}L\_2(x,y)\right) -$$

$$-a1(x,y)L\_1^3 + 3\,a2(x,y)L\_1^2L\_2 - 3\,a3(x,y)L\_1\,L\_2^2 + a4(x,y)L\_2^3,$$

where

$$L1 = \frac{\partial^2}{\partial y^2}a4(x,y) + 3\left(\frac{\partial}{\partial y}a2(x,y)\right)a4(x,y) + 3\,a2(x,y)\frac{\partial}{\partial y}a4(x,y) - 2\,\frac{\partial^2}{\partial x\partial y}a3(x,y) + \frac{\partial^2}{\partial x^2}a2(x,y) -$$

$$-2\,a4(x,y)\frac{\partial}{\partial x}a1(x,y) - a1(x,y)\frac{\partial}{\partial x}a4(x,y) - 3\,a3(x,y)\left(2\,\frac{\partial}{\partial y}a3(x,y) - \frac{\partial}{\partial x}a2(x,y)\right)$$

and

$$L2 = \frac{\partial^2}{\partial x^2}a1(x,y) - 3\left(\frac{\partial}{\partial x}a1(x,y)\right)a3(x,y) - 3\,a1(x,y)\frac{\partial}{\partial x}a3(x,y) + \frac{\partial^2}{\partial y^2}a3(x,y) - 2\,\frac{\partial^2}{\partial x\partial y}a2(x,y) +$$

$$+\left(\frac{\partial}{\partial y}a1(x,y)\right)a4(x,y) + 2\,a1(x,y)\frac{\partial}{\partial y}a4(x,y) - 3\,a2(x,y)\left(\frac{\partial}{\partial y}a3(x,y) - 2\,\frac{\partial}{\partial x}a2(x,y)\right).$$

As example, in the Lorenz case

$$Nu\_5 = Ax^2 + \frac{B}{x^2y^2} + C,$$

$$A = \alpha\,\beta(10\,\alpha - \alpha^2 - 6\,\delta),\quad B = \alpha(4/9\,\alpha^2 + 2/3\,\alpha\,\delta - 2\,\delta^2),$$
$$C = \alpha\,\left(2/9\,\alpha^4 + 6\,\epsilon\,\delta - 4\,\alpha\,\epsilon - \gamma\,\alpha^2\right),$$

where

$$\alpha = \frac{b+\sigma+1}{\sigma},\ \ \beta = \frac{1}{\sigma^2},\ \ \gamma = \frac{b(\sigma+1)}{\sigma^2},\ \ \delta = \frac{(\sigma+1)}{\sigma},\ \ \epsilon = \frac{b(r-1)}{\sigma^2}.$$

4 Toroidal vortex in viscous liquid

Following system of equations

$$2\,\frac{d^2}{dx^2}U(x) + 2\,x\frac{d^3}{dx^3}U(x) - 2\,G(x) - U(x)\frac{d^2}{dx^2}U(x) + \left(\frac{d}{dx}U(x)\right)^2 = 0,$$

$$2\,x\frac{d^2}{dx^2}V(x) - U(x)\frac{d}{dx}V(x) + V(x)\frac{d}{dx}U(x) = 0,\quad 4\,x^2\frac{d}{dx}G(x) + (V(x))^2 = 0$$

describes axial symmetric motion of viscous liquid

$$V\_r = \frac{\mu\,U(x)}{r},\quad V\_z = -2\,\frac{\mu\,z\,\frac{d}{dx}U(x)}{R^2}\,V\_phi = \frac{\mu\,z\sqrt{2}V(x)}{Rr},\quad B(x) = -\frac{d}{dx}U(x) - 1/4\,\frac{(U(x))^2}{x}$$

$$P = P0 + 2\,\mu^2\left(B(x) - 2\,\frac{z^2G(x)}{R^2}\right)R^{-2},$$

where U, V, B, G**-are unknown functions of coordinate** $x = r^2/R$**. (S.N.Aristov,2001)**

Given system of equations admits the reduction at the second order ODE of the form

$$\frac{d^2}{dx^2}y(x) + a1(x,y)\left(\frac{d}{dx}y(x)\right)^3 + 3\,a2(x,y)\left(\frac{d}{dx}y(x)\right)^2 + 3\,a3(x,y)\frac{d}{dx}y(x) + a4(x,y) = 0$$

with the coefficients

$$a1(x,y)=0,\quad a2(x,y)=-1/3\,\frac{3\,yx+2}{y\,(yx+1)},$$

$$a2(x,y)=-1/3\,\frac{y\,(x-1)\,(yx-1)}{yx+1},$$

$$a4(x,y)=1/3\,\frac{y^2\left(y+2\,yx+x^3y^3+2\,x^2y^2+yx^2+2\,y^2x^3+2\,x^4y^3+3\right)}{yx+1}$$

and so it properties may be investigated by the help of Liouville-Tresse-Cartan invariants

What is of corresponding dynamical system

$$\dot y=M(x,y,z),\ \ \dot z=N(x,y,z),\ \ \dot x=L(x,y,z)?$$

is open question.

5 2D-Hydrodynamic

Let us consider the equation describing $2D$-flows of incompressible liquids

$$\frac{\partial}{\partial t}\Delta(x,y,z,t)-\left(\frac{\partial}{\partial x}\psi(x,y,z,t)\right)\frac{\partial}{\partial y}\Delta(x,y,z,t)+\left(\frac{\partial}{\partial y}\psi(x,y,z,t)\right)\frac{\partial}{\partial x}\Delta(x,y,z,t)-$$

$$-\mu\left(\frac{\partial^2}{\partial x^2}\Delta(x,y,z,t)+\frac{\partial^2}{\partial y^2}\Delta(x,y,z,t)\right)=0,$$

where

$$\Delta(x,y,z,t)=\frac{\partial^2}{\partial x^2}\psi(x,y,z,t)+\frac{\partial^2}{\partial y^2}\psi(x,y,z,t),$$

and

$$U(x,y,z,t)=\frac{\partial}{\partial y}\psi(x,y,t),\quad V(x,y,z,t)=-\frac{\partial}{\partial x}\psi(x,y,t),\quad W(x,y,z,t)=0.$$

In particular case we get the p.d.e.

$$\left(\frac{\partial}{\partial x}A(x,y)\right)\frac{\partial^3}{\partial x^2\partial y}A(x,y)+\left(\frac{\partial}{\partial x}A(x,y)\right)\frac{\partial^3}{\partial y^3}A(x,y)-\left(\frac{\partial}{\partial y}A(x,y)\right)\frac{\partial^3}{\partial x^3}A(x,y)-\left(\frac{\partial}{\partial y}A(x,y)\right)\frac{\partial^3}{\partial y\partial x\partial y}A(x,y)+$$

$$+\mu\left(\frac{\partial^4}{\partial x^4}A(x,y)+2\,\frac{\partial^4}{\partial y\partial x^2\partial y}A(x,y)+\frac{\partial^4}{\partial y^4}A(x,y)\right)+=0.$$

after transformations of the form

$$\frac{\partial}{\partial y}A(x,y)=\frac{\frac{\partial}{\partial \tau}u(x,\tau)}{\frac{\partial}{\partial \tau}v(x,\tau)},\qquad \frac{\partial}{\partial x}A(x,y)=\frac{\partial}{\partial x}u(x,\tau)-\frac{\left(\frac{\partial}{\partial x}v(x,\tau)\right)\frac{\partial}{\partial \tau}u(x,\tau)}{\frac{\partial}{\partial \tau}v(x,\tau)}$$

and their extensions to higher order derivatives $A_{xx},\ \ A_{xy},\ \ A_{yy}$ and so on... we obtain an indefinite partial differential equation with respect to the functions

$$F(x,y,u_x,u_y...v_x,v_y...)=0,$$

with the help of which particular solutions of the original equation for the function $A(x,y)$ are constructed.

6 New exampel of flow

Following exact solution of the NS-equations

$$P(x,y,z,t) = -1/2\,z^4 - \_C3\,z^3 + \left(-5\,y^2 + 4\,x^2 - \_C6 - 1/2\,\_C3^2 + 3\right)z^2 +$$
$$+ \left(4\,\_C3\,x^2 - 5\,\_C3\,y^2 - \_C6\,\_C3 - 2\,\_C7\,y + 3\,\_C3\right)z - 9/2\,y^4 +$$
$$+ \left(-20\,x^2 - 1/2\,\_C3^2 - 3\,\_C6 + 9\right)y^2 + (-\_C7\,\_C3 +$$
$$+8\,x\alpha 2y - 8\,x^4 + (4\,\_C6 - 12)\,x^2 - 1/2\,\alpha 2^2 + \_F3\,(t)\,,$$
$$U(x,y,z,t) = \_C1\,y - 8\,xy + \_C2,$$
$$V(x,y,z,t) = \_C1\,x + \_C3\,z - 4\,x^2 + 3\,y^2 + z^2 + \_C9 - 3,$$
$$W(x,y,z,t) = 2\,zy + \_C3y + \beta,$$

describing the flow of incompressible liquid with parameters may be used to the further consideration...

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