Gradually Varied Flow in a Trapezoidal Channel

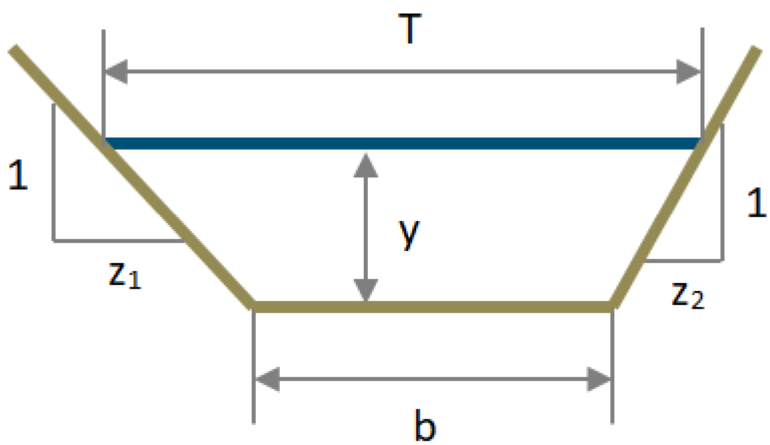
Water flows along a gently sloped trapezoidal channel with a known initial depth. As the flow progresses along the channel, the water depth eventually reaches a uniform depth that no longer changes with distance along the channel. This is known as the normal depth.

This is known as gradually varied flow. The governing differential equation is

$$\frac{d}{dx}(y) = \frac{S_0 - S_f}{1 - Fr^2}$$

where S0 is the channel slope, Sf is the energy gradient (or the rate at which energy is lost via friction) as given by the Manning formula, Fr is the Froude number, and x is the distance along the channel.

The normal depth (i.e. when y(x) no longer changes with x) is given by $S_0 - S_f$.



Given a trapezoidal channel, this application derives formulae for Sf and Fr. These are substituted into the differential equation, which is then numerically solved. Finally, the water surface profile along the channel is plotted.

Trapezoidal channels have several advantages over rectangular channels. Primarily, the wetted perimiter is small compared to the flow area; this reduces energy losses due to viscous drag, and thus maximizes flow.

Theory

Width of water surface $T := b + y(x) \cdot (z_1 + z_2)$

Cross-sectional area of flow $A := \frac{y(x)}{2} \cdot (b + T) = 0.500 \cdot y(x) \cdot (6 + 5 \cdot y(x))$

Wetted Perimeter

$$P := b + y(x) \cdot \left(\sqrt{1 + z_1^{\,2}} \cdot \sqrt{1 + z_2^{\,2}} \right)$$

Hydraulic radius

$$H := \frac{A}{P} = \frac{0.500 \cdot y(x) \cdot (6 + 5 \cdot y(x))}{3 + 7.071 \cdot y(x)}$$

Water velocity

$$v := \frac{Q}{A} = \frac{0.400}{y(x) \cdot (6 + 5 \cdot y(x))}$$

Froude number

$$Fr := \frac{v}{\sqrt{g \cdot y(x)}} = \frac{0.128}{y(x)^{3/2} \cdot (6 + 5 \cdot y(x))}$$

Slope of the energy gradient

$$S_f := \left(\frac{n \cdot v}{u \cdot H^{\frac{2}{3}}} \right)^2$$

$$S_f = \frac{2.520 \times 10^{-4}}{y(x)^2 \cdot (6 + 5 \cdot y(x))^2 \cdot \left(\frac{y(x) \cdot (6 + 5 \cdot y(x))}{3 + 7.071 \cdot y(x)} \right)^{4/3}}$$

Hence the complete differential equation is

$$de := \frac{d}{dx} y(x) = \frac{S_0 - S_f}{1 - Fr^2}$$

$$de = \frac{d}{dx}\, y(x) = \frac{0.001 - \dfrac{2.520 \times 10^{-4}}{y(x)^2 \cdot (6 + 5 \cdot y(x))^2 \cdot \left(\dfrac{y(x) \cdot (6 + 5 \cdot y(x))}{3 + 7.071 \cdot y(x)} \right)^{4/3}}}{1 - \dfrac{0.016}{y(x)^3 \cdot (6 + 5 \cdot y(x))^2}}$$

Parameters

Bottom width $b := 3$

Slope of channel sides $z_1 := 2$ $\quad z_2 := 3$

Channel slope $S_0 := 0.001$

Flowrate $Q := 0.2$

Water depth at maximum channel length. This will be the boundary condition on the differential equation. $y_0 := 0.8$

Channel roughness $n := 0.025$

Coefficient in Manning equation $u := 1$

Gravitational constant $g := 9.81$

Maximum channel length $L := 1000$

The differential equation reduces to

$$de = \frac{d}{dx}\, y(x) = \frac{0.001 - \dfrac{2.520 \times 10^{-4}}{y(x)^2 \cdot (6 + 5 \cdot y(x))^2 \cdot \left(\dfrac{y(x) \cdot (6 + 5 \cdot y(x))}{3 + 7.071 \cdot y(x)}\right)^{4/3}}}{1 - \dfrac{0.016}{y(x)^3 \cdot (6 + 5 \cdot y(x))^2}}$$

Critical Depth and Normal Depth

Normal depth $y_n := \mathrm{fsolve}(\mathrm{eval}(S_f = S_0, y(x) = y), y = 5) = 0.171$

Critical depth $y_c := \mathrm{fsolve}(\mathrm{subs}(y(x) = y, Fr = 1), y = 5) = 0.074$

The relative values of yn and yc determine the flow profile.

Numerical Solution of the Differential Equation

$res := \mathrm{dsolve}(\{de, y(L) = y_0\}, numeric)$

plots:-odeplot(res, x = 0 ..L, color = "DarkBlue", thickness = 4, view = [default, 0 ..1], title = "Water Surface Profile in a Trapezoidal Channel", titlefont = [Arial, 12]) =