# General solution option for PDEs and new methods for solving PDEs with Boundary Conditions

New options in pdsolve for users to ask for a general solution to PDEs and to know whether a solution from pdsolve is general. Also, many more partial differential equations with boundary condition (PDE and BC) problems can now be solved.

## New userinfo and *generalsolution* option in pdsolve

For a PDE of order N in 1 unknown depending on M independent variables, a *general solution* involves N arbitrary functions of M-1 arguments. Using differential algebra techniques, we have extended pdsolve's capabilities to identify a general solution for DE systems, even when the system involves ODEs and PDEs, algebraic equations, inequations, and/or mathematical functions.

The examples below show the new *generalsolution* option, as well as a new userinfo that displays whether a solution that is returned is or is not a general solution. The examples are all of differential equation systems but the same userinfo and *generalsolution* option work as well in the case of a single PDE.

**Example 1.**
Solve the determining PDE system for the infinitesimals of the symmetry generator of example 11 from Kamke's book. Tell whether the solution computed is a general solution.

> $restart : infolevel[pdsolve] := 3$

$$infolevel_{pdsolve} := 3 \tag{1.1}$$

The PDE system satisfied by the symmetries of Kamke's ODE example number 11 is

> $sys_1 := \left[\frac{\partial^2}{\partial y^2}\xi(x,y)=0, \frac{\partial^2}{\partial y^2}\eta(x,y) - 2\left(\frac{\partial^2}{\partial y \partial x}\xi(x,y)\right)=0, 3\,x^r y^n\left(\frac{\partial}{\partial y}\xi(x,y)\right)a + \left(2\left(\frac{\partial^2}{\partial y\partial x}\eta(x,y)\right)\right) - \frac{\partial^2}{\partial x^2}\xi(x,y)=0, 2\left(\frac{\partial}{\partial x}\xi(x,y)\right)x^r y^n a - x^r y^n\left(\frac{\partial}{\partial y}\eta(x,y)\right)a + \frac{\eta(x,y)\,a\,x^r y^n\,n}{y} + \frac{\xi(x,y)\,a\,x^r\,r\,y^n}{x} + \frac{\partial^2}{\partial x^2}\eta(x,y)=0\right]:$

This is a second order linear PDE system, with two unknowns $\{\eta(x,y), \xi(x,y)\}$ and four equations. Its *general solution* is given by the following, where we now can tell that the solution is a general one by reading the last line of the userinfo. Note that because the system is overdetermined, a general solution in this case *does not involve any arbitrary function*

> $sol_1 := pdsolve(sys_1)$

```
-> Solving ordering for the dependent variables of the PDE
system: [xi(x,y), eta(x,y)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [x, y]
tackling triangularized subsystem with respect to xi(x,y)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
tackling triangularized subsystem with respect to eta(x,y)
<- Returning a *general* solution
```

$$sol_1 := \left\{ \eta(x, y) = -\frac{\_C1\, y\, (r+2)}{n-1}, \xi(x, y) = \_C1\, x \right\} \tag{1.2}$$

Next we indicate to pdsolve that $n$ and $r$ are parameters of the problem, and that we want a solution for $n \neq 1$, making more difficult to identify by eye whether the solution returned is a general one. Again the last line of the userinfo indicates that pdsolve's solution is indeed a general one

> $sys_{1.1} := [op(sys_1), n \neq 1]$

$$sys_{1.1} := \left[ \frac{\partial^2}{\partial y^2} \xi(x, y) = 0, \frac{\partial^2}{\partial y^2} \eta(x, y) - 2 \left( \frac{\partial^2}{\partial x \partial y} \xi(x, y) \right) = 0, 3\, x^r y^n \left( \frac{\partial}{\partial y} \xi(x, y) \right) a + 2 \left( \frac{\partial^2}{\partial x \partial y} \eta(x, y) \right) - \left( \frac{\partial^2}{\partial x^2} \xi(x, y) \right) = 0, 2 \left( \frac{\partial}{\partial x} \xi(x, y) \right) x^r y^n a - x^r y^n \left( \frac{\partial}{\partial y} \eta(x, y) \right) a + \frac{\eta(x, y)\, a\, x^r y^n n}{y} + \frac{\xi(x, y)\, a\, x^r r\, y^n}{x} + \frac{\partial^2}{\partial x^2} \eta(x, y) = 0, n \neq 1 \right] \tag{1.3}$$

> $sol_{1.1} := pdsolve(sys_{1.1}, parameters = \{n, r\})$

```
-> Solving ordering for the dependent variables of the PDE
system: [r, n, xi(x,y), eta(x,y)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [x, y]
tackling triangularized subsystem with respect to r
tackling triangularized subsystem with respect to n
tackling triangularized subsystem with respect to xi(x,y)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
tackling triangularized subsystem with respect to eta(x,y)
tackling triangularized subsystem with respect to r
tackling triangularized subsystem with respect to n
tackling triangularized subsystem with respect to xi(x,y)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
tackling triangularized subsystem with respect to eta(x,y)
tackling triangularized subsystem with respect to r
tackling triangularized subsystem with respect to n
tackling triangularized subsystem with respect to xi(x,y)
First set of solution methods (general or quasi general
```

```
solution)
Trying simple case of a single derivative.
First set of solution methods successful
tackling triangularized subsystem with respect to eta(x,y)
tackling triangularized subsystem with respect to n
tackling triangularized subsystem with respect to xi(x,y)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
tackling triangularized subsystem with respect to eta(x,y)
tackling triangularized subsystem with respect to n
tackling triangularized subsystem with respect to xi(x,y)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
tackling triangularized subsystem with respect to eta(x,y)
tackling triangularized subsystem with respect to xi(x,y)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
tackling triangularized subsystem with respect to eta(x,y)
<- Returning a *general* solution
```

$$sol_{1.1} := \left\{n=2, r=-5, \eta(x,y)=y\,(\_C2\,x+3\,\_C1), \xi(x,y)=x\,(\_C2\,x+\_C1)\right\}, \left\{n=2, r=-\frac{20}{7}, \eta(x,y)=-\frac{2\,(-6\,\_C2\,x^2-98\,x^{8/7}\,\_C2\,a\,y-147\,\_C1\,a\,x\,y)}{343\,x\,a}, \xi(x,y)=\_C1\,x+\_C2\,x^{8/7}\right\}, \left\{n=2, r=-\frac{15}{7}, \eta(x,y)=-\frac{-49\,\_C1\,a\,x\,y-147\,x^{6/7}\,\_C2\,a\,y+12\,\_C2\,x}{343\,x\,a}, \xi(x,y)=\_C1\,x+\_C2\,x^{6/7}\right\}, \{n=2, r=r, \eta(x,y)=-\_C1\,y\,(r+2), \xi(x,y)=\_C1\,x\}, \left\{n=-r-3, r=r, \eta(x,y)=\frac{(4\,\_C2\,x+2\,\_C1+(\_C2\,x+\_C1)\,r)\,y}{r+4}, \xi(x,y)=x\,(\_C2\,x+\_C1)\right\}, \left\{n=n, r=r, \eta(x,y)=-\frac{\_C1\,y\,(r+2)}{n-1}, \xi(x,y)=\_C1\,x\right\} \quad \textbf{(1.4)}$$

> $map(pdetest, [sol_{1.1}], sys_{1.1})$

$$[[0,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,0]] \quad \textbf{(1.5)}$$

**Example 2.**
Compute the solution of the following (linear) overdetermined system involving two PDEs, three unknown functions, one of which depends on 2 variables and the other two depend on only 1 variable.

> $sys_2 := \left[-\left(\frac{\partial^2}{\partial r^2} F(r,s)\right) + \frac{\partial^2}{\partial s^2} F(r,s) + \frac{d}{dr} H(r) + \frac{d}{ds} G(s) + s = 0, \frac{\partial^2}{\partial r^2} F(r,s)\right.$

$$+\left(2\left(\frac{\partial^2}{\partial r\partial s} F(r,s)\right)\right)+\frac{\partial^2}{\partial s^2} F(r,s)-\frac{d}{dr} H(r)+\frac{d}{ds} G(s)-r=0\Bigg]:$$

The solution for the unknowns G, H, is given by the following expression, where again determining whether this solution, that depends on 3 arbitrary functions, $\_F1(s)$, $\_F2(r)$, $\_F3(s-r)$, is or is not a general solution, is non-obvious.

> $sol_2 := pdsolve(sys_2)$

```
-> Solving ordering for the dependent variables of the PDE
system: [F(r,s), H(r), G(s)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [r, s]
tackling triangularized subsystem with respect to F(r,s)
First set of solution methods (general or quasi general
solution)
Trying differential factorization for linear PDEs ...
differential factorization successful.
First set of solution methods successful
tackling triangularized subsystem with respect to H(r)
tackling triangularized subsystem with respect to G(s)
<- Returning a *general* solution
```

$$sol_2 := \left\{F(r,s)=\_F1(s)+\_F2(r)+\_F3(s-r)-\frac{r^2(r-3s)}{12}, G(s)=-\frac{s^2}{4}-\left(\frac{d}{ds}\_F1(s)\right)+\_C2, H(r)=-\frac{r^2}{4}+\frac{d}{dr}\_F2(r)+\_C1\right\} \quad \textbf{(1.6)}$$

> $pdetest(sol_2, sys_2)$

$$[0, 0] \quad \textbf{(1.7)}$$

**Example 3.**

Compute the solution of the following nonlinear system, consisting of Burger's equation and a possible potential.

> $$sys_3 := \left[\frac{\partial}{\partial t} u(x,t)+\left(2\,u(x,t)\left(\frac{\partial}{\partial x} u(x,t)\right)\right)-\frac{\partial^2}{\partial x^2} u(x,t)=0,\ \frac{\partial}{\partial t} v(x,t)=-\left(v(x,t)\left(\frac{\partial}{\partial x} u(x,t)\right)\right)+\left(v(x,t)\,u(x,t)^2\right),\ \frac{\partial}{\partial x} v(x,t)=-(u(x,t)\,v(x,t))\right]:$$

We see that in this case the solution returned *is not a general solution* but two particular ones; again the information is in the last line of the userinfo displayed

> $sol_3 := pdsolve(sys_3, [u, v])$

```
-> Solving ordering for the dependent variables of the PDE
system: [v(x,t), u(x,t)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [x, t]
tackling triangularized subsystem with respect to v(x,t)
tackling triangularized subsystem with respect to u(x,t)
First set of solution methods (general or quasi general
solution)
Second set of solution methods (complete solutions)
Trying methods for second order PDEs
Third set of solution methods (simple HINTs for separating
variables)
```

```
PDE linear in highest derivatives - trying a separation of
variables by *
HINT = *
Fourth set of solution methods
Trying methods for second order linear PDEs
Preparing a solution HINT ...
Trying HINT = _F1(x)*_F2(t)
Fourth set of solution methods
Preparing a solution HINT ...
Trying HINT = _F1(x)+_F2(t)
Trying travelling wave solutions as power series in tanh ...
* Using tau = tanh(t*C[2]+x*C[1]+C[0])
* Equivalent ODE system: {C[1]^2*(tau^2-1)^2*diff(diff(u(tau),
tau),tau)+(2*C[1]^2*(tau^2-1)*tau+C[2]*(tau^2-1)+2*u(tau)*C[1]*
(tau^2-1))*diff(u(tau),tau)}
* Ordering for functions: [u(tau)]
* Cases for the upper bounds: [[n[1] = 1]]
* Power series solution [1]: {u(tau) = tau*A[1,1]+A[1,0]}
* Solution [1] for {A[i, j], C[k]}: [[A[1,1] = 0], [A[1,0] =
-1/2*C[2]/C[1], A[1,1] = -C[1]]]
travelling wave solutions successful.
tackling triangularized subsystem with respect to v(x,t)
First set of solution methods (general or quasi general
solution)
Trying differential factorization for linear PDEs ...
Trying methods for PDEs "missing the dependent variable" ...
Second set of solution methods (complete solutions)
Trying methods for second order PDEs
Third set of solution methods (simple HINTs for separating
variables)
PDE linear in highest derivatives - trying a separation of
variables by *
HINT = *
Fourth set of solution methods
Trying methods for second order linear PDEs
Preparing a solution HINT ...
Trying HINT = _F1(x)*_F2(t)
Third set of solution methods successful
tackling triangularized subsystem with respect to u(x,t)
<- Returning a solution that *is not the most general one*
```

$$sol_3 := \left\{u(x,t) = -\_C2\tanh(\_C2\,x + \_C3\,t + \_C1) - \frac{\_C3}{2\,\_C2},\ v(x,t) = 0\right\},\ \left\{u(x,t) = -\frac{\sqrt{\_c_1}\left(\left(e^{\sqrt{\_c_1}\,x}\right)^2\_C1 - \_C2\right)}{\left(e^{\sqrt{\_c_1}\,x}\right)^2\_C1 + \_C2},\ v(x,t) = \_C3\,e^{\_c_1 t}\,\_C1\,e^{\sqrt{\_c_1}\,x} + \frac{\_C3\,e^{\_c_1 t}\,\_C2}{e^{\sqrt{\_c_1}\,x}}\right\} \quad \textbf{(1.8)}$$

> $map(pdetest, [sol_3], sys_3)$

$$[[0, 0, 0], [0, 0, 0]] \quad \textbf{(1.9)}$$

This example is also good for illustrating the other related new feature: one can now request to pdsolve to *only compute a general solution* (it will return NULL if it cannot achieve that). Turn OFF userinfos and try with this example

> $infolevel[pdsolve] := 1:$

This returns NULL:

> $pdsolve(sys_3, [u, v], generalsolution)$

**Example 4.**

Another where the solution returned is particular, this time for a linear system, conformed by 38 PDEs, also from differential equation symmetry analysis

$$
\begin{aligned}
> sys_4 := \Bigg[ & \frac{\partial}{\partial u}\,\xi_1(x, y, z, t, u) = 0,\ \frac{\partial}{\partial x}\,\xi_1(x, y, z, t, u) - \frac{\partial}{\partial y}\,\xi_2(x, y, z, t, u) = 0,\ \frac{\partial}{\partial u}\,\xi_2(x, y, z, t, u) = 0,\\
& -\left(\frac{\partial}{\partial y}\,\xi_1(x, y, z, t, u)\right) - \frac{\partial}{\partial x}\,\xi_2(x, y, z, t, u) = 0,\ \frac{\partial}{\partial u}\,\xi_3(x, y, z, t, u) = 0,\ \frac{\partial}{\partial x}\,\xi_1(x, y, z, t, u) - \frac{\partial}{\partial z}\,\xi_3(x, y, z, t, u) = 0,\\
& -\left(\frac{\partial}{\partial y}\,\xi_3(x, y, z, t, u)\right) - \frac{\partial}{\partial z}\,\xi_2(x, y, z, t, u) = 0,\ -\left(\frac{\partial}{\partial z}\,\xi_1(x, y, z, t, u)\right) - \frac{\partial}{\partial x}\,\xi_3(x, y, z, t, u) = 0,\\
& \frac{\partial}{\partial u}\,\xi_4(x, y, z, t, u) = 0,\ \frac{\partial}{\partial t}\,\xi_3(x, y, z, t, u) - \frac{\partial}{\partial z}\,\xi_4(x, y, z, t, u) = 0,\ \frac{\partial}{\partial t}\,\xi_2(x, y, z, t, u) - \frac{\partial}{\partial y}\,\xi_4(x, y, z, t, u) = 0,\\
& \frac{\partial}{\partial t}\,\xi_1(x, y, z, t, u) - \frac{\partial}{\partial x}\,\xi_4(x, y, z, t, u) = 0,\ -\left(\frac{\partial}{\partial x}\,\xi_1(x, y, z, t, u)\right) + \frac{\partial}{\partial t}\,\xi_4(x, y, z, t, u) = 0,\\
& \frac{\partial^2}{\partial y^2}\,\eta_1(x, y, z, t, u) + \frac{\partial^2}{\partial z^2}\,\eta_1(x, y, z, t, u) - \frac{\partial^2}{\partial t^2}\,\eta_1(x, y, z, t, u) + \frac{\partial^2}{\partial x^2}\,\eta_1(x, y, z, t, u) = 0,\\
& \frac{\partial^2}{\partial u^2}\,\eta_1(x, y, z, t, u) = 0,\ \frac{\partial^2}{\partial u \partial x}\,\eta_1(x, y, z, t, u) + \frac{\partial^2}{\partial x^2}\,\xi_1(x, y, z, t, u) = 0,\\
& \frac{\partial^2}{\partial x \partial y}\,\xi_1(x, y, z, t, u) + \frac{\partial^2}{\partial u \partial y}\,\eta_1(x, y, z, t, u) = 0,\ -\left(\frac{\partial^2}{\partial y^2}\,\xi_1(x, y, z, t, u)\right) + \frac{\partial^2}{\partial u \partial x}\,\eta_1(x, y, z, t, u) = 0,\\
& \frac{\partial^2}{\partial x \partial z}\,\xi_1(x, y, z, t, u) + \frac{\partial^2}{\partial u \partial z}\,\eta_1(x, y, z, t, u) = 0,\ \frac{\partial^2}{\partial y \partial z}\,\xi_1(x, y, z, t, u) = 0,\\
& -\left(\frac{\partial^2}{\partial z^2}\,\xi_1(x, y, z, t, u)\right) + \frac{\partial^2}{\partial u \partial x}\,\eta_1(x, y, z, t, u) = 0,\ -\left(\frac{\partial^2}{\partial t \partial u}\,\eta_1(x, y, z, t, u)\right) - \frac{\partial^2}{\partial t \partial x}\,\xi_1(x, y, z, t, u) = 0,\\
& \frac{\partial^2}{\partial t \partial y}\,\xi_1(x, y, z, t, u) = 0,\ \frac{\partial^2}{\partial t \partial z}\,\xi_1(x, y, z, t, u) = 0,\ \frac{\partial^2}{\partial t^2}\,\xi_1(x, y, z, t, u) + \frac{\partial^2}{\partial u \partial x}\,\eta_1(x, y, z, t, u) = 0,\\
& -\left(\frac{\partial^2}{\partial z^2}\,\xi_2(x, y, z, t, u)\right) + \frac{\partial^2}{\partial u \partial y}\,\eta_1(x, y, z, t, u) = 0,\ \frac{\partial^2}{\partial t \partial z}\,\xi_2(x, y, z, t, u) = 0,\\
& \frac{\partial^2}{\partial t^2}\,\xi_2(x, y, z, t, u) + \frac{\partial^2}{\partial u \partial y}\,\eta_1(x, y, z, t, u) = 0,\ \frac{\partial^2}{\partial t^2}\,\xi_3(x, y, z, t, u) + \frac{\partial^2}{\partial u \partial z}\,\eta_1(x, y, z, t, u) = 0,\\
& \frac{\partial^3}{\partial u \partial x^2}\,\eta_1(x, y, z, t, u) = 0,\ \frac{\partial^3}{\partial u \partial x \partial y}\,\eta_1(x, y, z, t, u) = 0,\ \frac{\partial^3}{\partial u \partial y^2}\,\eta_1(x, y, z, t, u) = 0,\\
& \frac{\partial^3}{\partial u \partial x \partial z}\,\eta_1(x, y, z, t, u) = 0,\ \frac{\partial^3}{\partial u \partial y \partial z}\,\eta_1(x, y, z, t, u) = 0,\ \frac{\partial^3}{\partial u \partial z^2}\,\eta_1(x, y, z, t, u) = 0,\\
& \frac{\partial^3}{\partial t \partial u \partial x}\,\eta_1(x, y, z, t, u) = 0,\ \frac{\partial^3}{\partial t \partial u \partial y}\,\eta_1(x, y, z, t, u) = 0,\ \frac{\partial^3}{\partial t \partial u \partial z}\,\eta_1(x, y, z, t, u) = 0 \Bigg]:
\end{aligned}
$$

There are 38 coupled equations

> $nops(sys_4)$

$$38 \tag{1.10}$$

When requesting *a general solution* pdsolve returns NULL:

> $pdsolve(sys_4, generalsolution)$

A solution that is not a general one, is however computed by default if calling pdsolve without the *generalsolution* option. In this case again the last line of the userinfo indicates that the solution returned is not a general solution

> $infolevel[pdsolve] := 3$

$$infolevel_{pdsolve} := 3 \tag{1.11}$$

> $sol_4 := pdsolve(sys_4)$

```
-> Solving ordering for the dependent variables of the PDE
system: [eta[1](x,y,z,t,u), xi[1](x,y,z,t,u), xi[2](x,y,z,t,u),
xi[3](x,y,z,t,u), xi[4](x,y,z,t,u)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [t, x, y, z, u]
tackling triangularized subsystem with respect to eta[1](x,y,z,
t,u)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
-> Solving ordering for the dependent variables of the PDE
system: [_F1(x,y,z,t), _F2(x,y,z,t)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [t, x, y, z, u]
tackling triangularized subsystem with respect to _F1(x,y,z,t)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
-> Solving ordering for the dependent variables of the PDE
system: [_F3(x,y,z), _F4(x,y,z)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [x, y, z, t]
tackling triangularized subsystem with respect to _F3(x,y,z)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
tackling triangularized subsystem with respect to _F4(x,y,z)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
-> Solving ordering for the dependent variables of the PDE
system: [_F5(y,z), _F6(y,z)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [y, z, x]
```

```
tackling triangularized subsystem with respect to _F5(y,z)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
tackling triangularized subsystem with respect to _F6(y,z)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
-> Solving ordering for the dependent variables of the PDE
system: [_F7(z), _F8(z)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [z, y]
tackling triangularized subsystem with respect to _F7(z)
tackling triangularized subsystem with respect to _F8(z)
tackling triangularized subsystem with respect to _F2(x,y,z,t)
First set of solution methods (general or quasi general
solution)
Trying differential factorization for linear PDEs ...
Trying methods for PDEs "missing the dependent variable" ...
Second set of solution methods (complete solutions)
Third set of solution methods (simple HINTs for separating
variables)
PDE linear in highest derivatives - trying a separation of
variables by *
HINT = *
Fourth set of solution methods
Preparing a solution HINT ...
Trying HINT = _F3(x)*_F4(y)*_F5(z)*_F6(t)
Third set of solution methods successful
tackling triangularized subsystem with respect to xi[1](x,y,z,
t,u)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
-> Solving ordering for the dependent variables of the PDE
system: [_F1(x,z,t), _F2(x,z,t)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [t, x, z, y]
tackling triangularized subsystem with respect to _F1(x,z,t)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
tackling triangularized subsystem with respect to _F2(x,z,t)
First set of solution methods (general or quasi general
solution)
```

```
Trying simple case of a single derivative.
First set of solution methods successful
-> Solving ordering for the dependent variables of the PDE
system: [_F3(x,t), _F4(x,t)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [t, x, z]
tackling triangularized subsystem with respect to _F3(x,t)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
tackling triangularized subsystem with respect to _F4(x,t)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
-> Solving ordering for the dependent variables of the PDE
system: [_F5(x), _F6(x)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [x, t]
tackling triangularized subsystem with respect to _F5(x)
tackling triangularized subsystem with respect to _F6(x)
tackling triangularized subsystem with respect to xi[2](x,y,z,
t,u)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
-> Solving ordering for the dependent variables of the PDE
system: [_F1(t), _F2(t)]
-> Solving ordering for the independent variables (can be
changed using the ivars option): [t, z]
tackling triangularized subsystem with respect to _F1(t)
tackling triangularized subsystem with respect to _F2(t)
tackling triangularized subsystem with respect to xi[3](x,y,z,
t,u)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
First set of solution methods (general or quasi general
```

```
solution)
Trying simple case of a single derivative.
First set of solution methods successful
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
tackling triangularized subsystem with respect to xi[4](x,y,z,
t,u)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
First set of solution methods successful
<- Returning a solution that *is not the most general one*
```

$$sol_4 := \Bigg\{\eta_1(x, y, z, t, u) = \frac{1}{e^{\sqrt{-\_c_1}\, x}\, e^{\sqrt{-\_c_2}\, y}\, e^{\sqrt{-\_c_3}\, z}} \Big(\_C13 \Big(\_C10 \left(e^{\sqrt{-\_c_3}\, z}\right)^2 + \_C11\Big) \Big(\_C8 \left(e^{\sqrt{-\_c_2}\, y}\right)^2 + \_C9\Big) \Big(\_C6 \left(e^{\sqrt{-\_c_1}\, x}\right)^2 + \_C7\Big) \cos\left(\sqrt{-\_c_1 - \_c_2 - \_c_3}\; t\right) + \_C12 \Big(\_C10 \left(e^{\sqrt{-\_c_3}\, z}\right)^2 + \_C11\Big) \Big(\_C8 \left(e^{\sqrt{-\_c_2}\, y}\right)^2 + \_C9\Big) \Big(\_C6 \left(e^{\sqrt{-\_c_1}\, x}\right)^2 + \_C7\Big) \sin\left(\sqrt{-\_c_1 - \_c_2 - \_c_3}\; t\right) + u\, e^{\sqrt{-\_c_1}\, x}\, e^{\sqrt{-\_c_2}\, y}\, e^{\sqrt{-\_c_3}\, z} (\_C1\, t + \_C2\, x + \_C3\, y + \_C4\, z + \_C5)\Big), \xi_1(x, y, z, t, u) = -\frac{\_C2\, x^2}{2} + \frac{(-2\,\_C1\, t - 2\,\_C3\, y - 2\,\_C4\, z + 2\,\_C17)\, x}{2} + \frac{(-t^2 + y^2 + z^2)\,\_C2}{2} + \_C16\, t + \_C15\, z + \_C14\, y + \_C18, \xi_2(x, y, z, t, u) = -\frac{\_C3\, y^2}{2} + \frac{(-2\,\_C1\, t - 2\,\_C2\, x - 2\,\_C4\, z + 2\,\_C17)\, y}{2} + \frac{(-t^2 + x^2 + z^2)\,\_C3}{2} + \_C20\, t \tag{1.12}$$

$$+ \_C19\, z - \_C14\, x + \_C21, \xi_3(x, y, z, t, u) = -\frac{\_C4\, z^2}{2} + \frac{(-2\,\_C1\, t - 2\,\_C2\, x - 2\,\_C3\, y + 2\,\_C17)\, z}{2} + \frac{(-t^2 + x^2 + y^2)\,\_C4}{2} + \_C22\, t - \_C19\, y - \_C15\, x + \_C23, \xi_4(x, y, z, t, u) = -\frac{\_C1\, t^2}{2} + \frac{(-2\,\_C2\, x - 2\,\_C3\, y - 2\,\_C4\, z + 2\,\_C17)\, t}{2} + \frac{(-x^2 - y^2 - z^2)\,\_C1}{2} + \_C20\, y + \_C22\, z + \_C16\, x + \_C24 \Big\}$$

> $pdetest(sol_4, sys_4)$

[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] **(1.13)**

**Example 5.**

Finally, the new userinfos also indicates whether a solution is a general solution when working with PDEs that involve anticommutative variables set using the Physics package

> $with(Physics, Setup)$

$$[Setup] \quad \textbf{(1.14)}$$

Set first $\theta$ and $Q$ as suffixes for variables of type/anticommutative

> $Setup(anticommutativepre = \{Q, \theta\})$

* Partial match of 'anticommutativepre' against keyword 'anticommutativeprefix'

$$[anticommutativeprefix = \{Q, \_\lambda, \theta\}] \quad \textbf{(1.15)}$$

A PDE system example with two unknown anticommutative functions of four variables, two commutative and two anticommutative; to avoid redundant typing in the input that follows and redundant display of information on the screen let's use PDEtools:-diff_table and PDEtools:-declare

> $PDEtools\text{:-}declare(Q(x, y, \theta_1, \theta_2))$

$$Q(x, y, \theta_1, \theta_2) \text{ will now be displayed as } Q \quad \textbf{(1.16)}$$

> $q := PDEtools\text{:-}diff\_table(Q(x, y, \theta_1, \theta_2))$

$$q := \mathrm{table}(symmetric, Physics/diff, [() = Q]) \quad \textbf{(1.17)}$$

Consider the system formed by these two PDEs (because of the $q$ diff_table just defined, we can enter derivatives directly using the function's name indexed by the differentiation variables)

> $pde_1 := q_{x, y, \theta_1} + q_{x, y, \theta_2} - q_{y, \theta_1, \theta_2} = 0$

$$pde_1 := Q_{y, x, \theta_1} + Q_{y, x, \theta_2} - Q_{\theta_1, y, \theta_2} = 0 \quad \textbf{(1.18)}$$

> $pde_2 := q_{\theta_1} = 0$

$$pde_2 := Q_{\theta_1} = 0 \quad \textbf{(1.19)}$$

The solution returned for this system is indeed a *general solution*

> $pdsolve([pde_1, pde_2])$

```
-> Solving ordering for the dependent variables of the PDE
system: [_F4(x,y), _F2(x,y), _F3(x,y)]
-> Solving ordering for the independent variables (can be
```

```
changed using the ivars option): [x, y]
tackling triangularized subsystem with respect to _F4(x,y)
tackling triangularized subsystem with respect to _F2(x,y)
tackling triangularized subsystem with respect to _F3(x,y)
First set of solution methods (general or quasi general
solution)
Trying simple case of a single derivative.
HINT = _F6(x)+_F5(y)
Trying HINT = _F6(x)+_F5(y)
HINT is successful
First set of solution methods successful
<- Returning a *general* solution
```

$$Q = \_F1(x, y)\ \_\lambda 1 + (\_F6(x) + \_F5(y))\ \theta_2 \tag{1.20}$$

This solution involves an *anticommutative constant* $\_\lambda 2$, analogous to the commutative constants $\_Cn$ where n is an integer.

# PDE&BC in semi-infinite domains for which a bounded solution is sought can now also be solved via Laplace transforms

Maple is now able to solve more PDE&BC problems via Laplace transforms.
Laplace transforms act to change derivatives with respect to one of the independent variables of the domain into multiplication operations in the transformed domain. After applying a Laplace transform to the original problem, we can simplify the problem using the transformed BC, then solve the problem in the transformed domain, and finally apply the inverse Laplace transform to arrive at the final solution. It is important to remember to give pdsolve any necessary restrictions on the variables and constants of the problem, by means of the "assuming" command.

A new feature is that we can now tell pdsolve that the dependent variable is bounded, by means of the optional argument HINT = boundedseries.

> *restart* :

Consider the problem of a falling cable lying on a table that is suddenly removed (cf. David J. Logan's *Applied Partial Differential Equations* p.115).

> $pde_1 := \frac{\partial^2}{\partial t^2} u(x, t) = c^2 \left( \frac{\partial^2}{\partial x^2} u(x, t) \right) - g :$

$iv_1 := u(x, 0) = 0, u(0, t) = 0, D_2(u)(x, 0) = 0 :$

If we ask pdsolve to solve this problem without the condition of boundedness of the solution, we obtain:

> $pdsolve([pde_1, iv_1])$ **assuming** $0 < t, 0 < x, 0 < c$

$$u(x, t) = \frac{1}{2 c^2} \left( g\,(c\,t - x)^2\, \theta\left(\frac{c\,t - x}{c}\right) - c^2 \left( g\,t^2 - 2\, invlaplace\left(e^{\frac{s\,x}{c}}\ \_F1(s), s, t\right) + 2\, invlaplace\left(e^{-\frac{s\,x}{c}}\ \_F1(s), s, t\right) \right) \right) \tag{2.1}$$

If we now ask for a bounded solution, by means of the option HINT = boundedseries, pdsolve simplifies the problem accordingly.

> $ans_1 := pdsolve\left(\left[pde_1, iv_1\right], HINT = boundedseries\right)$ **assuming** $0 < t, 0 < x, 0 < c$

$$ans_1 := u(x,t) = \frac{g\left(\theta\left(t - \frac{x}{c}\right)(c\,t - x)^2 - c^2 t^2\right)}{2\,c^2} \tag{2.2}$$

And we can check this answer against the original problem, if desired:

> $pdetest\left(ans_1, \left[pde_1, iv_1\right]\right)$ **assuming** $0 < t, 0 < x, 0 < c$

$$[0, 0, 0, 0] \tag{2.3}$$

## How it works, step by step

Let us see the process this problem is solved by pdsolve, step by step.

First, the Laplace transform is applied to the PDE:

> $with(inttrans):$

> $transformed\_PDE := laplace\left((lhs - rhs)\left(pde_1\right), t, s\right)$

$$transformed\_PDE := s^2\, laplace(u(x,t), t, s) - \mathrm{D}_2(u)(x,0) - s\,u(x,0) - c^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} laplace(u(x,t), t, s) + \frac{g}{s} \tag{2.1.1}$$

and the result is simplified using the initial conditions:

> $simplified\_transformed\_PDE := eval\left(transformed\_PDE, \left\{iv_1\right\}\right)$

$$simplified\_transformed\_PDE := s^2\, laplace(u(x,t), t, s) - c^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} laplace(u(x,t), t, s) + \frac{g}{s} \tag{2.1.2}$$

Next, we call the function "laplace(u(x,t),t,s)" by the new name U:

> $eq\_U := subs(laplace(u(x,t), t, s) = U(x,s), simplified\_transformed\_PDE)$

$$eq\_U := s^2\, U(x,s) - c^2 \left(\frac{\partial^2}{\partial x^2} U(x,s)\right) + \frac{g}{s} \tag{2.1.3}$$

And this equation, which is really an ODE, is solved:

> $solution\_U := dsolve(eq\_U, U(x,s))$

$$solution\_U := U(x,s) = \mathrm{e}^{\frac{s\,x}{c}}\, \_F2(s) + \mathrm{e}^{-\frac{s\,x}{c}}\, \_F1(s) - \frac{g}{s^3} \tag{2.1.4}$$

Now, since we want a BOUNDED solution, the term with the positive exponential must be zero, and we are left with:

> $bounded\_solution\_U := subs\left(coeff\left(rhs(solution\_U), \mathrm{e}^{\frac{s \cdot x}{c}}\right) = 0, solution\_U\right)$

$$bounded\_solution\_U := U(x,s) = \mathrm{e}^{-\frac{s\,x}{c}}\, \_F1(s) - \frac{g}{s^3} \tag{2.1.5}$$

Now, the initial solution must also be satisfied. Here it is, in the transformed domain:

> $Laplace\_BC := laplace(u(0, t), t, s) = 0$

$$Laplace\_BC := laplace(u(0, t), t, s) = 0 \quad \textbf{(2.1.6)}$$

Or, in the new variable U,

> $Laplace\_BC\_U := U(0, s) = 0$

$$Laplace\_BC\_U := U(0, s) = 0 \quad \textbf{(2.1.7)}$$

And by applying it to *bounded_solution_U*, we find the relationship

> $simplify(subs(x = 0, rhs(bounded\_solution\_U))) = 0$

$$\frac{\_F1(s)\, s^3 - g}{s^3} = 0 \quad \textbf{(2.1.8)}$$

> $isolate(\textbf{(2.1.8)}, indets(\textbf{(2.1.8)}, unknown)[1])$

$$\_F1(s) = \frac{g}{s^3} \quad \textbf{(2.1.9)}$$

so that our solution now becomes

> $bounded\_solution\_U := subs(\textbf{(2.1.9)}, bounded\_solution\_U)$

$$bounded\_solution\_U := U(x, s) = \frac{\mathrm{e}^{-\frac{s\,x}{c}}\, g}{s^3} - \frac{g}{s^3} \quad \textbf{(2.1.10)}$$

to which we now apply the inverse Laplace transform to obtain the solution to the problem:

> $u(x, t) = invlaplace(rhs(bounded\_solution\_U), s, t)$ **assuming** $0 < x, 0 < t, 0 < c$

$$u(x, t) = \frac{g\left(-t^2 + \frac{\theta\left(t - \frac{x}{c}\right)(c\,t - x)^2}{c^2}\right)}{2} \quad \textbf{(2.1.11)}$$

# Four other related examples

A few other examples:

> $pde_2 := \frac{\partial^2}{\partial t^2} u(x, t) = c^2 \left(\frac{\partial^2}{\partial x^2} u(x, t)\right):$

$iv_2 := u(x, 0) = 0, u(0, t) = g(t), \mathrm{D}_2(u)(x, 0) = 0:$

> $ans_2 := pdsolve([pde_2, iv_2], HINT = boundedseries)$ **assuming** $0 < t, 0 < x, 0 < c$

$$ans_2 := u(x, t) = \theta\left(t - \frac{x}{c}\right) g\left(\frac{c\,t - x}{c}\right) \quad \textbf{(2.2.1)}$$

> $pdetest(ans_2, [pde_2, iv_2])$ **assuming** $0 < t, 0 < x, 0 < c$

$$[0, 0, 0, 0] \quad \textbf{(2.2.2)}$$

> $pde_3 := \frac{\partial}{\partial t} u(x, t) = k\left(\frac{\partial^2}{\partial x^2} u(x, t)\right) : iv_3 := u(x, 0) = 0, u(0, t) = 1 :$

> $ans_3 := pdsolve([pde_3, iv_3], HINT = boundedseries)$ **assuming** $0 < t, 0 < x, 0 < k;$

$$ans_3 := u(x, t) = 1 - \mathrm{erf}\left(\frac{x}{2\sqrt{t}\,\sqrt{k}}\right) \quad \textbf{(2.2.3)}$$

> $pdetest\left(ans_3, \left[pde_3, \left(iv_3\right)_2\right]\right)$

$$[0, 0] \tag{2.2.4}$$

> $pde_4 := \frac{\partial}{\partial t} u(x, t) = k \left( \frac{\partial^2}{\partial x^2} u(x, t) \right) : iv_4 := u(x, 0) = \mu, u(0, t) = \lambda :$

> $ans_4 := pdsolve\left(\left[pde_4, iv_4\right], HINT = boundedseries\right)$ **assuming** $0 < t, 0 < x, 0 < k$

$$ans_4 := u(x, t) = (-\lambda + \mu)\, \mathrm{erf}\left( \frac{x}{2\sqrt{t}\sqrt{k}} \right) + \lambda \tag{2.2.5}$$

> $pdetest\left(ans_4, \left[pde_4, \left(iv_4\right)_2\right]\right)$

$$[0, 0] \tag{2.2.6}$$

The following is an example from David J. Logan's *Applied Partial Differential Equations* p.76:

> $pde_5 := \frac{\partial}{\partial t} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) :$
> $iv_5 := u(x, 0) = 0, u(0, t) = f(t) :$

> $ans_5 := pdsolve\left(\left[pde_5, iv_5\right], HINT = boundedseries\right)$ **assuming** $0 < t, 0 < x$

$$ans_5 := u(x, t) = \frac{x \left( \int_0^t \frac{f(\_U1)\, \mathrm{e}^{-\frac{x^2}{4t - 4\_U1}}}{(t - \_U1)^{3/2}} \, \mathrm{d}\_U1 \right)}{2\sqrt{\pi}} \tag{2.2.7}$$

# More PDE&BC problems in bounded spatial domains can now be solved via eigenfunction (Fourier) expansions

The code for solving PDE&BC problems in bounded spatial domains has been expanded. The method works by separating the variables by product, so that the problem is transformed into an ODE system (with initial and/or boundary conditions), and for one of the variables it is a Sturm-Liouville problem (a type of eigenvalue problem) which has infinitely many solutions - hence the infinite series representation of the solutions.

> *restart* :

Here is a simple example for the heat equation:

> $pde_6 := \frac{\partial}{\partial t} u(x, t) = k \left( \frac{\partial^2}{\partial x^2} u(x, t) \right) :$
> $iv_6 := u(0, t) = 0, u(l, t) = 0 :$

> $ans_6 := pdsolve\left(\left[pde_6, iv_6\right]\right)$ **assuming** $0 < l$;

$$ans_6 := u(x, t) = \sum_{\_Z1 = 1}^{\infty} \_C1 \sin\left( \frac{\_Z1\, \pi\, x}{l} \right) \mathrm{e}^{-\frac{k \pi^2 \_Z1^2 t}{l^2}} \tag{3.1}$$

> pdetest( $ans_6$, [$pde_6$, $iv_6$]);

$$[0, 0, 0] \quad \textbf{(3.2)}$$

Now, consider the displacements of a string governed by the wave equation, where c is a constant (cf. Logan p.28).

> $pde_7 := \frac{\partial^2}{\partial t^2} u(x,t) = c^2 \left( \frac{\partial^2}{\partial x^2} u(x,t) \right)$ :
 $iv_7 := u(0,t) = 0, u(l,t) = 0$ :

> $ans_7$ := pdsolve([$pde_7$, $iv_7$]) **assuming** $0 < l$;

$$ans_7 := u(x,t) = \sum_{\_Z2=1}^{\infty} \sin\left(\frac{\_Z2\,\pi\,x}{l}\right)\left(\sin\left(\frac{c\,\_Z2\,\pi\,t}{l}\right)\_C1 + \cos\left(\frac{c\,\_Z2\,\pi\,t}{l}\right)\_C5\right) \quad \textbf{(3.3)}$$

> pdetest($ans_7$, [$pde_7$, $iv_7$]);

$$[0, 0, 0] \quad \textbf{(3.4)}$$

Another wave equation problem (cf. Logan p.130):

> $pde_8 := \frac{\partial^2}{\partial t^2} u(x,t) - c^2 \left( \frac{\partial^2}{\partial x^2} u(x,t) \right) = 0$ :
 $iv_8 := u(0,t) = 0, D_2(u)(x,0) = 0, D_1(u)(l,t) = 0, u(x,0) = f(x)$ :

> $ans_8$ := (pdsolve([$pde_8$, $iv_8$], $u(x,t)$)) **assuming** $0 \le x, x \le l$);

$$ans_8 := u(x,t) = \sum_{\_Z3=1}^{\infty} \frac{1}{l}\left(2\left(\int_0^l f(x)\sin\left(\frac{\pi(1+2\_Z3)x}{2l}\right) dx\right)\sin\left(\frac{\pi(1+2\_Z3)x}{2l}\right)\cos\left(\frac{c\,\pi(1+2\_Z3)t}{2l}\right)\right) \quad \textbf{(3.5)}$$

> pdetest($ans_8$, [$pde_8$, $iv_8[1..3]$]);

$$[0, 0, 0, 0] \quad \textbf{(3.6)}$$

Here is a problem with periodic boundary conditions (cf. Logan p.131). The function $u(x,t)$ stands for the concentration of a chemical dissolved in water within a tubular ring of circumference $2\,l$. The initial concentration is given by $f(x)$, and the variable $x$ is the arc-length parameter that varies from 0 to $2\,l$.

> $pde_9 := \frac{\partial}{\partial t} u(x,t) = M \left( \frac{\partial^2}{\partial x^2} u(x,t) \right)$ :
 $iv_9 := u(0,t) = u(2\,l,t), D_1(u)(0,t) = D_1(u)(2\,l,t), u(x,0) = f(x)$ :

> $ans_9$ := pdsolve([$pde_9$, $iv_9$], $u(x,t)$) **assuming** $0 \le x, x \le 2\,l$;

$$ans_9 := u(x,t) = \frac{\_C8}{2} + \left(\sum_{\_Z5=1}^{\infty} \frac{1}{l}\left(\left(\left(\left(\int_0^{2l} f(x)\sin\left(\frac{\_Z5\,\pi\,x}{l}\right) dx\right)\sin\left(\frac{\_Z5\,\pi\,x}{l}\right) + \left( \right.\right.\right.\right.\right. \quad \textbf{(3.7)}$$

$$\int_0^{2l} f(x)\cos\left(\frac{\_Z5\,\pi\,x}{l}\right)\mathrm{d}x\Bigg)\cos\left(\frac{\_Z5\,\pi\,x}{l}\right)\Bigg)\,\mathrm{e}^{-\frac{M\pi^2\,\_Z5^2\,t}{l^2}}\Bigg)\Bigg)$$

> $pdetest(ans_9, [pde_9, iv_9[1..2]]);$

$$[0, 0, 0] \tag{3.8}$$

The following problem is for heat flow with both boundaries insulated (cf. Logan p.166, 3rd edition)

> $pde_{10} := \frac{\partial}{\partial t}\, u(x,t) = k\left(\frac{\partial^2}{\partial x^2}\, u(x,t)\right):$
> $iv_{10} := \mathrm{D}_1(u)(0,t) = 0,\ \mathrm{D}_1(u)(l,t) = 0,\ u(x,0) = f(x):$

> $ans_{10} := pdsolve([pde_{10}, iv_{10}], u(x,t))$ **assuming** $0 \le x, x \le l;$

$$ans_{10} := u(x,t) = \sum_{\_Z7=1}^{\infty} \frac{2\left(\int_0^l f(x)\cos\left(\frac{\_Z7\,\pi\,x}{l}\right)\mathrm{d}x\right)\cos\left(\frac{\_Z7\,\pi\,x}{l}\right)\mathrm{e}^{-\frac{k\pi^2\,\_Z7^2\,t}{l^2}}}{l} \tag{3.9}$$

> $pdetest(ans_{10}, [pde_{10}, iv_{10}[1..2]]);$

$$[0, 0, 0] \tag{3.10}$$

This is a problem in a bounded domain with the presence of a source. A source term represents an outside influence in the system and leads to an inhomogeneous PDE (cf. Logan p.149):

> $pde_{11} := \frac{\partial^2}{\partial t^2}\, u(x,t) - c^2\left(\frac{\partial^2}{\partial x^2}\, u(x,t)\right) = p(x,t):$
> $iv_{11} := u(0,t) = 0,\ u(\pi,t) = 0,\ u(x,0) = 0,\ \mathrm{D}_2(u)(x,0) = 0:$

> $ans_{11} := pdsolve([pde_{11}, iv_{11}], u(x,t));$

$$ans_{11} := u(x,t) = \tag{3.11}$$

$$\int_0^t \left(\sum_{\_Z8=1}^{\infty} \frac{2\left(\int_0^{\pi} p(x,\tau 1)\sin(\_Z8\,x)\,\mathrm{d}x\right)\sin(\_Z8\,x)\sin(c\,\_Z8\,(t-\tau 1))}{\pi\,\_Z8\,c}\right)\mathrm{d}\tau 1$$

Current pdetest is unable to verify that this solution cancels the $pde_{11}$ mainly because it currently fails in identifying that there is a fourier expansion in it, but its subroutines for testing the boundary conditions work well with this problem

> $pdetest\_BC := `pdetest/BC`:$
> $pdetest\_BC(\{ans_{11}\}, [iv_{11}], [u(x,t)]);$

$$[0, 0, 0, 0] \tag{3.12}$$

Consider a heat absorption-radiation problem in the bounded domain $0 \le x \le 2, t \ge 0$:

> $pde_{12} := \frac{\partial}{\partial t}\, u(x,t) = \frac{\partial^2}{\partial x^2}\, u(x,t):$

$iv_{12} := u(x,0) = f(x), D_1(u)(0,t) + u(0,t) = 0, D_1(u)(2,t) + u(2,t) = 0:$

> $ans_{12} := pdsolve\left(\left[pde_{12}, iv_{12}\right], u(x,t)\right)$ **assuming** $0 \le x$ **and** $x \le 2, 0 \le t$;

$$ans_{12} := u(x,t) = \frac{\_C8}{2} + \left(\sum_{\_Z9=1}^{\infty}\left(\left(\left(\int_0^2 f(x)\sin\left(\frac{\_Z9\,\pi\,x}{2}\right)dx\right)\sin\left(\frac{\_Z9\,\pi\,x}{2}\right) + \left(\int_0^2 f(x)\cos\left(\frac{\_Z9\,\pi\,x}{2}\right)dx\right)\cos\left(\frac{\_Z9\,\pi\,x}{2}\right)\right)e^{-\frac{\pi^2\,\_Z9^2\,t}{4}}\right)\right) \tag{3.13}$$

> $pdetest\left(ans_{12}, pde_{12}\right)$;

$$0 \tag{3.14}$$

Consider the nonhomogeneous wave equation problem (cf. Logan p.213, 3rd edition):

> $pde_{13} := \frac{\partial^2}{\partial t^2}u(x,t) = A\,x + \frac{\partial^2}{\partial x^2}u(x,t):$

$iv_{13} := u(0,t) = 0, u(1,t) = 0, u(x,0) = 0, D_2(u)(x,0) = 0:$

> $ans_{13} := pdsolve\left(\left[pde_{13}, iv_{13}\right]\right)$;

$$ans_{13} := u(x,t) = \int_0^t \left(\sum_{\_Z10=1}^{\infty} \frac{2\,A\left(\int_0^1 x\sin(\pi\,\_Z10\,x)\,dx\right)\sin(\pi\,\_Z10\,x)\sin(\pi\,\_Z10\,(t-\tau 1))}{\pi\,\_Z10}\right)d\tau 1 \tag{3.15}$$

> $pdetest\_BC\left(\left\{ans_{13}\right\}, \left[iv_{13}\right], [u(x,t)]\right)$;

$$[0, 0, 0, 0] \tag{3.16}$$

Consider the following Schrödinger equation with zero potential energy (cf. Logan p.30):

> $pde_{14} := I\,h\left(\frac{\partial}{\partial t}f(x,t)\right) = -\frac{h^2\left(\frac{\partial^2}{\partial x^2}f(x,t)\right)}{(2\,m)}:$

$iv_{14} := f(0,t) = 0, f(d,t) = 0:$

> $ans_{14} := pdsolve\left(\left[pde_{14}, iv_{14}\right]\right)$ **assuming** $0 < d$;

$$ans_{14} := f(x,t) = \sum_{\_Z11=1}^{\infty}\_C1\sin\left(\frac{\_Z11\,\pi\,x}{d}\right)e^{\frac{-\frac{I}{2}\,h\,\pi^2\,\_Z11^2\,t}{d^2\,m}} \tag{3.17}$$

> $pdetest\left(ans_{14}, \left[pde_{14}, iv_{14}\right]\right)$;

$$[0, 0, 0] \tag{3.18}$$

# Another method for linear PDE&BC with spatial initial conditions

This method is for problems of the form

$$\frac{\partial}{\partial t} w = M_w, \quad w(x_i, 0) = f(x_i)$$

or

$$\frac{\partial^2}{\partial t^2} w = M_w, \quad w(x_i, 0) = f(x_i), \quad \left.\left(\frac{\partial}{\partial t} w\right)\right|_{t=0} = g(x_i)$$

where $M$ is an arbitrary linear differential operator of any order which only depends on the spatial variables $(x_i)$.

Here are some examples:

> $pde_{15} := \frac{\partial}{\partial t} w(x1, x2, x3, t) - \left(\frac{\partial^2}{\partial x2\, \partial x1} w(x1, x2, x3, t)\right) - \left(\frac{\partial^2}{\partial x3\, \partial x1} w(x1, x2, x3, t)\right) - \left(\frac{\partial^2}{\partial x3^2} w(x1, x2, x3, t)\right) + \frac{\partial^2}{\partial x3\, \partial x2} w(x1, x2, x3, t) = 0 :$
> $iv_{15} := w(x1, x2, x3, 0) = x1^5\, x2\, x3 :$

> $pdsolve([pde_{15}, iv_{15}]);$

$$w(x1, x2, x3, t) = 20\, x1^3 \left(\left(\frac{x2\, x3}{20} - \frac{t}{20}\right) x1^2 + \frac{t\,(x2 + x3)\, x1}{4} + t^2\right) \quad \textbf{(4.1)}$$

> $pdetest(\%, [pde_{15}, iv_{15}]);$

$$[0, 0] \quad \textbf{(4.2)}$$

Here are two examples for which the derivative with respect to t is of the second order, and two initial conditions are given:

> $pde_{16} := \frac{\partial^2}{\partial t^2} w(x1, x2, x3, t) = \frac{\partial^2}{\partial x2\, \partial x1} w(x1, x2, x3, t) + \frac{\partial^2}{\partial x3\, \partial x1} w(x1, x2, x3, t) + \frac{\partial^2}{\partial x3^2} w(x1, x2, x3, t) - \left(\frac{\partial^2}{\partial x3\, \partial x2} w(x1, x2, x3, t)\right) :$
> $iv_{16} := w(x1, x2, x3, 0) = x1^3\, x2^2 + x3, \mathrm{D}_4(w)(x1, x2, x3, 0) = -(x2\, x3) + x1 :$

> $pdsolve([pde_{16}, iv_{16}]);$

$$w(x1, x2, x3, t) = x1^3\, x2^2 + x3 - t\, x2\, x3 + t\, x1 + 3\, t^2\, x1^2\, x2 + \frac{1}{6}\, t^3 + \frac{1}{2}\, t^4\, x1 \quad \textbf{(4.3)}$$

> $pdetest(\%, [pde_{16}, iv_{16}]);$

$$[0, 0, 0] \quad \textbf{(4.4)}$$

> $pde_{17} := \frac{\partial^2}{\partial t^2} w(x1, x2, x3, t) = \frac{\partial^2}{\partial x2\, \partial x1} w(x1, x2, x3, t) + \frac{\partial^2}{\partial x3\, \partial x1} w(x1, x2, x3, t) + \frac{\partial^2}{\partial x3^2} w(x1, x2, x3, t) - \left(\frac{\partial^2}{\partial x3\, \partial x2} w(x1, x2, x3, t)\right) :$

$iv_{17} := w(x1, x2, x3, 0) = x1^3\, x3^2 + \sin(x1),\ D_4(w)(x1, x2, x3, 0) = \cos(x1) - x2\, x3$ :

> $pdsolve([pde_{17}, iv_{17}]);$

$$w(x1, x2, x3, t) = \frac{t^4\, x1}{2} + t^2\, x1^3 + 3\, t^2\, x1^2\, x3 + x1^3\, x3^2 + \frac{t^3}{6} - t\, x2\, x3 + \cos(x1)\, t + \sin(x1) \quad \textbf{(4.5)}$$

> $pdetest(\%, [pde_{17}, iv_{17}]);$

$$[0, 0, 0] \quad \textbf{(4.6)}$$

# More PDE&BC problems solved via first finding the PDE's general solution.

The following are examples of PDE&BC problems for which pdsolve is successful in first calculating the PDE's general solution, and then fitting the initial or boundary condition to it.

> $pde_{18} := \frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) = 0$ :

$iv_{18} := u(0, y) = \frac{\sin(y)}{y}$ :

If we ask pdsolve to solve the problem, we get:

> $ans_{18} := pdsolve([pde_{18}, iv_{18}]);$

$$ans_{18} := u(x, y) = \frac{\sin(-y + \mathrm{I}\, x) + \_F2(y - \mathrm{I}\, x)\,(y - \mathrm{I}\, x) + (-y + \mathrm{I}\, x)\,\_F2(y + \mathrm{I}\, x)}{-y + \mathrm{I}\, x} \quad \textbf{(5.1)}$$

and we can check this answer by using pdetest:

> $pdetest(ans_{18}, [pde_{18}, iv_{18}]);$

$$[0, 0] \quad \textbf{(5.2)}$$

## How it works, step by step:

The general solution for just the PDE is:

> $gensol := pdsolve(pde_{18});$

$$gensol := u(x, y) = \_F1(y - \mathrm{I}\, x) + \_F2(y + \mathrm{I}\, x) \quad \textbf{(5.1.1)}$$

Substituting in the condition $iv_{18}$, we get:

$$u(0, y) = \frac{\sin(y)}{y} \quad \textbf{(5.1.2)}$$

> $gensol\_with\_condition := eval(rhs(gensol), x = 0) = rhs(iv_{18});$

$$gensol\_with\_condition := \_F1(y) + \_F2(y) = \frac{\sin(y)}{y} \quad \textbf{(5.1.3)}$$

We then isolate one of the functions above (we can choose either one, in this case), convert it into a function operator, and then apply it to *gensol*

> $\_F1 = unapply(solve(\textbf{(5.1.3)}, \_F1(y)), y)$

$$\_F1 = \left(y \mapsto -\frac{\_F2(y)\, y - \sin(y)}{y}\right) \quad \textbf{(5.1.4)}$$

> $eval(gensol, \textbf{(5.1.4)})$

$$u(x, y) = -\frac{\_F2(y - \mathrm{I}\, x)\,(y - \mathrm{I}\, x) + \sin(-y + \mathrm{I}\, x)}{y - \mathrm{I}\, x} + \_F2(y + \mathrm{I}\, x) \quad \textbf{(5.1.5)}$$

### Three other related examples

> $pde_{19} := \frac{\partial^2}{\partial x^2} u(x, y) + \left(\frac{1}{2} \left(\frac{\partial^2}{\partial y^2} u(x, y)\right)\right) = 0:$

$iv_{19} := u(0, y) = \frac{\sin(y)}{y}:$

> $pdsolve([pde_{19}, iv_{19}]);$

$$u(x, y) = \frac{1}{I\sqrt{2}\,x - 2y}\left(2\sin\left(-y + \frac{I\sqrt{2}\,x}{2}\right) + \left(-I\sqrt{2}\,x + 2y\right)\_F2\left(y - \frac{I\sqrt{2}\,x}{2}\right) + \left(I\sqrt{2}\,x - 2y\right)\_F2\left(y + \frac{I\sqrt{2}\,x}{2}\right)\right) \quad \textbf{(5.2.1)}$$

> $pdetest(\%, [pde_{19}, iv_{19}]);$

$$[0, 0] \quad \textbf{(5.2.2)}$$

> $pde_{20} := \frac{\partial^2}{\partial x^2} u(x, y) + \left(\frac{1}{2} \left(\frac{\partial^2}{\partial y^2} u(x, y)\right)\right) = 0:$

$iv_{20} := u(x, 0) = \frac{\sin(x)}{x}:$

> $pdsolve([pde_{20}, iv_{20}]);$

$$u(x, y) = \frac{1}{I\sqrt{2}\,x - 2y}\left(\sinh\left(\frac{\sqrt{2}\left(I\sqrt{2}\,x - 2y\right)}{2}\right)\sqrt{2} - \left(I\sqrt{2}\,x - 2y\right)\left(\_F2\left(-y + \frac{I\sqrt{2}\,x}{2}\right) - \_F2\left(y + \frac{I\sqrt{2}\,x}{2}\right)\right)\right) \quad \textbf{(5.2.3)}$$

> $pdetest(\%, [pde_{20}, iv_{20}]);$

$$[0, 0] \quad \textbf{(5.2.4)}$$

> $pde_{21} := \frac{\partial^2}{\partial r^2} u(r, t) + \frac{\left(\frac{\partial}{\partial r} u(r, t)\right)}{r} + \frac{\left(\frac{\partial^2}{\partial t^2} u(r, t)\right)}{r^2} = 0:$

$iv_{21} := u(3, t) = \sin(6t):$

> $ans_{21} := pdsolve([pde_{21}, iv_{21}]);$

$$ans_{21} := u(r, t) = -\_F2(-2\,I\ln(3) + I\ln(r) + t) + \sin(-6\,I\ln(3) + 6\,I\ln(r) + 6t) + \_F2(-I\ln(r) + t) \quad \textbf{(5.2.5)}$$

> $pdetest(ans_{21}, [pde_{21}, iv_{21}]);$

$$[0, 0] \quad \textbf{(5.2.6)}$$

## More PDE&BC problems are now solved by using a Fourier transform.

> $restart:$

Consider the following problem with an initial condition:

> $pde_{22} := \frac{\partial}{\partial t} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) + m:$

$iv_{22} := u(x, 0) = \sin(x)$ :

pdsolve can solve this problem directly:

> $ans_{22} := pdsolve([pde_{22}, iv_{22}]);$

$$ans_{22} := u(x, t) = \sin(x)\, e^{-t} + m\, t \qquad \textbf{(6.1)}$$

And we can check this answer against the original problem, if desired:

> $pdetest(ans_{22}, [pde_{22}, iv_{22}]);$

$$[0, 0] \qquad \textbf{(6.2)}$$

## How it works, step by step

Similarly to the Laplace transform method, we start the solution process by first applying the Fourier transform to the PDE:

> $with(inttrans)$ :

> $transformed\_PDE := fourier((lhs - rhs)(pde_{22}) = 0, x, s);$

$$transformed\_PDE := -2\, m\, \pi\, \delta(s) + \frac{d}{dt}\, fourier(u(x, t), x, s) + s^2\, fourier(u(x, t), x, s) = 0 \qquad \textbf{(6.1.1)}$$

Next, we call the function "fourier(u(x,t),x,s1)" by the new name U:

> $transformed\_PDE\_U := subs(fourier(u(x, t), x, s) = U(t, s), transformed\_PDE);$

$$transformed\_PDE\_U := -2\, m\, \pi\, \delta(s) + \frac{\partial}{\partial t}\, U(t, s) + s^2\, U(t, s) = 0 \qquad \textbf{(6.1.2)}$$

And this equation, which is really an ODE, is solved:

> $solution\_U := dsolve(transformed\_PDE\_U, U(t, s));$

$$solution\_U := U(t, s) = (2\, m\, \pi\, \delta(s)\, t + \_F1(s))\, e^{-s^2 t} \qquad \textbf{(6.1.3)}$$

Now, we apply the Fourier transform to the initial condition $iv_{22}$:

$$u(x, 0) = \sin(x) \qquad \textbf{(6.1.4)}$$

> $transformed\_IC := fourier(iv_{22}, x, s);$

$$transformed\_IC := fourier(u(x, 0), x, s) = I\, \pi\, (-\delta(s - 1) + \delta(s + 1)) \qquad \textbf{(6.1.5)}$$

Or, in the new variable U,

> $transformed\_IC\_U := U(0, s) = rhs(transformed\_IC);$

$$transformed\_IC\_U := U(0, s) = I\, \pi\, (-\delta(s - 1) + \delta(s + 1)) \qquad \textbf{(6.1.6)}$$

Now, we evaluate solution_U at t = 0:

> $solution\_U\_at\_IC := eval(solution\_U, t = 0);$

$$solution\_U\_at\_IC := U(0, s) = \_F1(s) \qquad \textbf{(6.1.7)}$$

and substitute the transformed initial condition into it:

> $eval(solution\_U\_at\_IC, \{transformed\_IC\_U\});$

$$I\, \pi\, (-\delta(s - 1) + \delta(s + 1)) = \_F1(s) \qquad \textbf{(6.1.8)}$$

Putting this into our solution_U, we get

> $eval(solution\_U, \{(rhs = lhs)(\textbf{(6.1.8)})\});$

$$U(t, s) = (2\, m\, \pi\, \delta(s)\, t + I\, \pi\, (-\delta(s - 1) + \delta(s + 1)))\, e^{-s^2 t} \qquad \textbf{(6.1.9)}$$

Finally, we apply the inverse Fourier transformation to this,

> $solution := u(x, t) = invfourier(rhs(\mathbf{(6.1.9)}), s, x);$

$$solution := u(x, t) = \sin(x)\, e^{-t} + m\, t \qquad \mathbf{(6.1.10)}$$

# PDE&BC problems that used to require the option HINT = `+` to be solved are now solved automatically

The following are two examples of PDE&BC problems which used to require the option HINT = `+` in order to be solved. This is now done automatically within pdsolve.

> $pde_{23} := \frac{\partial^2}{\partial r^2} u(r, t) + \frac{\left(\frac{\partial}{\partial r} u(r, t)\right)}{r} + \frac{\left(\frac{\partial^2}{\partial t^2} u(r, t)\right)}{r^2} = 0 :$
> $iv_{23} := u(1, t) = 0, u(2, t) = 5 :$

> $ans_{23} := pdsolve([pde_{23}, iv_{23}]);$

$$ans_{23} := u(r, t) = \frac{5 \ln(r)}{\ln(2)} \qquad \mathbf{(7.1)}$$

> $pdetest(ans_{23}, [pde_{23}, iv_{23}]);$

$$[0, 0, 0] \qquad \mathbf{(7.2)}$$

> $pde_{24} := \frac{\partial^2}{\partial y^2} u(x, y) + \frac{\partial^2}{\partial x^2} u(x, y) = 6\, x - 6\, y :$

> $iv_{24} := u(x, 0) = x^3 + (11\, x) + 1, u(x, 2) = x^3 + (11\, x) - 7, u(0, y) = -y^3 + 1, u(4, y) = -y^3 + 109 :$

> $ans_{24} := pdsolve([pde_{24}, iv_{24}]);$

$$ans_{24} := u(x, y) = x^3 - y^3 + 11\, x + 1 \qquad \mathbf{(7.3)}$$

> $pdetest(ans_{24}, [pde_{24}, iv_{24}]);$

$$[0, 0, 0, 0, 0] \qquad \mathbf{(7.4)}$$