Physics

Maple provides a state-of-the-art environment for algebraic computations in Physics, with emphasis on ensuring that the computational experience is as natural as possible. The theme of the Physics project for Maple 2022 has been the consolidation of the functionality introduced in previous releases, including a significant speed-up across the package and significant enhancements in the areas of Particle Physics, Functional Differentiation in general relativity, and Integral Vector Calculus.

As part of its commitment to providing the best possible computational environment in Physics, Maplesoft launched a Maple Physics: Research and Development website in 2014, which enabled users to download research versions of the package, ask questions, and provide feedback. The results from this accelerated exchange have been incorporated into the Physics package in Maple 2022. The presentation below illustrates both the novelties and the kind of mathematical formulations that can now be performed.

The StandardModel package

StandardModel is a Physics's package that implements computational representations for the mathematical objects formulating the Standard Model in particle physics. The package includes field representations for the leptons and quarks of the model, as well as for Weinberg's angle, the Higgs boson, and the fields and field strengths after breaking symmetries and most of the fields before that. Loading the package sets things to proceed computing with the model.

```
> with(Physics) : with(StandardModel)
```

Setting lowercaselatin_is letters to represent Dirac spinor indices

Setting lowercaselatin_ah letters to represent SU(3) adjoint representation, (1..8) indices

Setting uppercaselatin_ah letters to represent SU(3) fundamental representation, (1..3) indices

Setting uppercaselatin_is letters to represent SU(2) adjoint representation, (1..3) indices

Setting uppercasegreek letters to represent SU(2) fundamental representation, (1..2) indices

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Defined as the electron, muon and tau leptons and corresponding neutrinos: e_j, μ_j, τ_j, $\nu^{(e)}{}_j$, $\nu^{(\mu)}{}_j$, $\nu^{(\tau)}{}_j$

Defined as the up, down, charm, strange, top and bottom quarks: $u_{j,A}$, $c_{j,A}$, $t_{j,A}$, $d_{j,A}$, $s_{j,A}$, $b_{j,A}$

Defined as gauge tensors: B_μ, $\mathbb{B}_{\mu,\nu}$, A_μ, $\mathbb{F}_{\mu,\nu}$, $W_{\mu,J}$, $\mathbb{W}_{\mu,\nu,J}$, $W^+{}_\mu$, $\mathbb{W}^+{}_{\mu,\nu}$, $W^-{}_\mu$, $\mathbb{W}^-{}_{\mu,\nu}$, Z_μ, $\mathbb{Z}_{\mu,\nu}$, $G_{\mu,a}$, $\mathbb{G}_{\mu,\nu,a}$

Defined as Gell-Mann (Glambda), Pauli (Psigma) and Dirac (Dgamma) matrices: λ_a, σ_J, γ_μ

Defined as the electric, weak and strong coupling constants: g_e, g_w, g_s

Defined as the charge for 1) the electron, muon and tauon, 2) the up, charm and top, and 3) the down, strange and bottom: $q_e = -1,\ q_u = \frac{2}{3},\ q_d = -\frac{1}{3}$

Defined as the weak isospin for 1) the electron, muon and tauon, 2) the up, charm and top, 3) the down, strange and bottom, and 4) all the neutrinos: $I_e = -\frac{1}{2},\ I_u = \frac{1}{2},\ I_d = -\frac{1}{2},\ I_n = \frac{1}{2},$

You can use the active form without the % prefix, or the 'value' command to give the corresponding value to any of the inert representations $q_e, q_u, q_d, I_e, I_u, I_d, I_n,$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (x, y, z, t)\}$

Minkowski spacetime with signatre $(- - - +)$

$[I_d, I_e, I_n, I_u, q_d, q_e, q_u,$ *BField, BFieldStrength, Bottom, CKM, Charm, Down, ElectromagneticField, ElectromagneticFieldStrength, Electron, ElectronNeutrino, FSU3, Glambda, GluonField, GluonFieldStrength, HiggsBoson, Lagrangian, Muon, MuonNeutrino, Strange, Tauon, TauonNeutrino, Top, Up, WField, WFieldStrength, WMinusField, WMinusFieldStrength, WPlusField, WPlusFieldStrength, WeinbergAngle, ZField, ZFieldStrength,* $g_e, g_s, g_w]$ **(1)**

The Leptons, Quarks, Gauge Fields and structure constants of the model

The massless fields of the model are the electromagnetic field **A**, the gluons **G** and neutrinos $\nu^{(\mu)}$, $\nu^{(\tau)}$ and $\nu^{(e)}$

> *Setup*(*massless*)

** Partial match of 'massless' against keyword 'masslessfields'*

$[masslessfields = \{\mathbf{G}, \nu^{(\mu)}, \nu^{(\tau)}, \mathbf{A}, \nu^{(e)}\}]$ **(2)**

The Leptons and Quarks of the model are

> *StandardModel*:-*Leptons*

$[\mathbf{e}, \mu, \tau, \nu^{(e)}, \nu^{(\mu)}, \nu^{(\tau)}]$ **(3)**

> *StandardModel*:-*Quarks*

$[\mathbf{u}, \mathbf{c}, \mathbf{t}, \mathbf{d}, \mathbf{s}, \mathbf{b}]$ **(4)**

The Gauge fields

> *StandardModel*:-*GaugeFields*

$$[\mathbf{A}, \mathbb{F}, \mathbf{B}, \mathbb{B}, \mathbf{W}, \mathbb{W}, \mathbf{G}, \mathbb{G}, \mathbf{W}^{-}, \mathbb{W}^{-}, \mathbf{W}^{+}, \mathbb{W}^{+}, \mathbf{Z}, \mathbb{Z}] \quad (5)$$

For readability, omit the functionality of all these fields from the display of formulas that follows (see CompactDisplay) and use the lowercase i instead of the uppercase I to represent the imaginary unit

```
> CompactDisplay(StandardModel:-Leptons(X), StandardModel:-Quarks(X), StandardModel:-
      GaugeFields(X), HiggsBoson(X), quiet)
> interface(imaginaryunit = i) :
```

The definitions of the gauge fields can be seen as with any other tensor of the Physics package using the keyword *definition*

```
> ElectromagneticField[definition]
```

$$\mathbf{A}_{\mu} = \sin(\boldsymbol{\theta}_{\mathbf{w}})\, \mathbf{W}_{\mu}^{\,3} + \cos(\boldsymbol{\theta}_{\mathbf{w}})\, \mathbf{B}_{\mu} \quad (6)$$

```
> map(u -> u[definition], StandardModel:-GaugeFields)
```

$$\left[\mathbf{A}_{\mu} = \sin(\boldsymbol{\theta}_{\mathbf{w}})\, \mathbf{W}_{\mu}^{\,3} + \cos(\boldsymbol{\theta}_{\mathbf{w}})\, \mathbf{B}_{\mu}, \mathbb{F}_{\mu,\nu} = \partial_{\mu}(\mathbf{A}_{\nu}) - \partial_{\nu}(\mathbf{A}_{\mu}), \mathbf{B}_{\mu} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix}, \mathbb{B}_{\mu,\nu} \right. \quad (7)$$

$$= \partial_{\mu}(\mathbf{B}_{\nu}) - \partial_{\nu}(\mathbf{B}_{\mu}), \mathbf{W}_{\mu,J} = \begin{bmatrix} \mathbf{W}_{1,1} & \mathbf{W}_{1,2} & \mathbf{W}_{1,3} \\ \mathbf{W}_{2,1} & \mathbf{W}_{2,2} & \mathbf{W}_{2,3} \\ \mathbf{W}_{3,1} & \mathbf{W}_{3,2} & \mathbf{W}_{3,3} \\ \mathbf{W}_{4,1} & \mathbf{W}_{4,2} & \mathbf{W}_{4,3} \end{bmatrix}, \mathbb{W}_{\mu,\nu,J} = \partial_{\mu}(\mathbf{W}_{\nu,J}) - \partial_{\nu}(\mathbf{W}_{\mu,J})$$

$$\left. + g_w\, \epsilon_{J,K,L}\, \mathbf{W}_{\mu,K}\, \mathbf{W}_{\nu,L}, \mathbf{G}_{\mu,a} = \begin{bmatrix} \mathbf{G}_{1,1} & \mathbf{G}_{1,2} & \mathbf{G}_{1,3} & \mathbf{G}_{1,4} & \mathbf{G}_{1,5} & \mathbf{G}_{1,6} & \mathbf{G}_{1,7} & \mathbf{G}_{1,8} \\ \mathbf{G}_{2,1} & \mathbf{G}_{2,2} & \mathbf{G}_{2,3} & \mathbf{G}_{2,4} & \mathbf{G}_{2,5} & \mathbf{G}_{2,6} & \mathbf{G}_{2,7} & \mathbf{G}_{2,8} \\ \mathbf{G}_{3,1} & \mathbf{G}_{3,2} & \mathbf{G}_{3,3} & \mathbf{G}_{3,4} & \mathbf{G}_{3,5} & \mathbf{G}_{3,6} & \mathbf{G}_{3,7} & \mathbf{G}_{3,8} \\ \mathbf{G}_{4,1} & \mathbf{G}_{4,2} & \mathbf{G}_{4,3} & \mathbf{G}_{4,4} & \mathbf{G}_{4,5} & \mathbf{G}_{4,6} & \mathbf{G}_{4,7} & \mathbf{G}_{4,8} \end{bmatrix}, \right.$$

$$\left. \mathbb{G}_{\mu,\nu,a} = \partial_{\mu}(\mathbf{G}_{\nu,a}) - \partial_{\nu}(\mathbf{G}_{\mu,a}) + g_s f_{su3_{a,b,c}}\, \mathbf{G}_{\mu,b}\, \mathbf{G}_{\nu,c}, \mathbf{W}^{-}{}_{\mu} = \frac{\left(\mathbf{W}_{\mu}^{\,1} + \mathrm{i}\, \mathbf{W}_{\mu}^{\,2}\right)\sqrt{2}}{2}, \right.$$

$$\mathbb{W}^{-}_{\mu,\nu} = \partial_\mu\left(\mathbf{W}^{-}_{\nu}\right) - \partial_\nu\left(\mathbf{W}^{-}_{\mu}\right),\ \mathbf{W}^{+}_{\mu} = \frac{\left(\mathbf{W}_{\mu}^{\ 1} - i\,\mathbf{W}_{\mu}^{\ 2}\right)\sqrt{2}}{2},\ \mathbb{W}^{+}_{\mu,\nu} = \partial_\mu\left(\mathbf{W}^{+}_{\nu}\right)$$

$$\left. - \partial_\nu\left(\mathbf{W}^{+}_{\mu}\right),\ \mathbf{Z}_{\mu} = \cos\left(\boldsymbol{\theta}_{\mathbf{w}}\right)\mathbf{W}_{\mu}^{\ 3} - \sin\left(\boldsymbol{\theta}_{\mathbf{w}}\right)\mathbf{B}_{\mu},\ \mathbb{Z}_{\mu,\nu} = \partial_\mu\left(\mathbf{Z}_{\nu}\right) - \partial_\nu\left(\mathbf{Z}_{\mu}\right)\right]$$

Note that the conventions used in the definitions of covariant derivatives (not shown above) and field strength tensors, follow Peskin, S. "An Introduction to Quantum Field Theory", also the Wikipedia, and are not uniform in the literature: the gauge term involving the gluon in the covariant derivative of the quarks, e.g. the Top, $\mathbf{u}_{j,A}$, has a minus sign and the third term in the gluon field strength definition (shown above) has a plus sign:

> $D\_[\mathrm{mu}](Up[j, A](X))$:
> $\%= expand(\%)$

$$\nabla_\mu\left(\mathbf{u}_{j,A}\right) = \partial_\mu\left(\mathbf{u}_{j,A}\right) - \frac{2\,i\,g_e\,\mathbf{u}_{j,A}\,\mathbf{A}_\mu}{3} - \frac{i\,g_s\left(\lambda_a\right)_{A,B}\,\mathbf{u}_{j,B}\,\mathbf{G}_{\mu,a}}{2} \tag{8}$$

> $GluonFieldStrength[definition]$

$$\mathbb{G}_{\mu,\nu,a} = \partial_\mu\left(\mathbf{G}_{\nu,a}\right) - \partial_\nu\left(\mathbf{G}_{\mu,a}\right) + g_s f_{su3_{a,b,c}}\,\mathbf{G}_{\mu,b}\,\mathbf{G}_{\nu,c} \tag{9}$$

The convention for the signs in the definitions of $\mathbf{A}_\mu$ and $\mathbf{Z}_\mu$ in **(7)** also follow Peskin's book and the presentation of the Standard Model in Wikipedia.

The Gell-Mann matrices, that enter gauge terms in the interaction Lagrangian of the StandardModel are represented by *Glambda*, implemented as a tensor with an SU(3) adjoint representation index, all of whose components are matrices

> $Glambda[\ ]$

$$\lambda_a = \left[\begin{array}{cccccccc} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 \end{array}\right] \tag{10}$$

> $seq(Glambda[a, matrix], a = 1..8)$

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \tag{11}$$

$$\lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix},\ \lambda_8 = \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & -\frac{2\sqrt{3}}{3} \end{bmatrix}$$

These matrices satisfy a SU(3) algebra

> *Library*:-*DefaultAlgebraRules*(*Glambda*)

$$\left[\lambda_b, \lambda_c\right]_{-} = 2\,\mathrm{i}\,f_{su3_{a,\,b,\,c}}\,\lambda_a \tag{12}$$

The structure constants $f_{su3_{a,\,b,\,c}}$ entering **(12)** and interaction Lagrangian terms of the StandardModel form a three-dimensional array of 8 x 8 matrices represented by the command *FSU3*. implemented as a tensor with three SU(3) adjoint representation indices. As with any other tensor of the Physics package, to see its components you can use the keyword *matrix*, e.g.

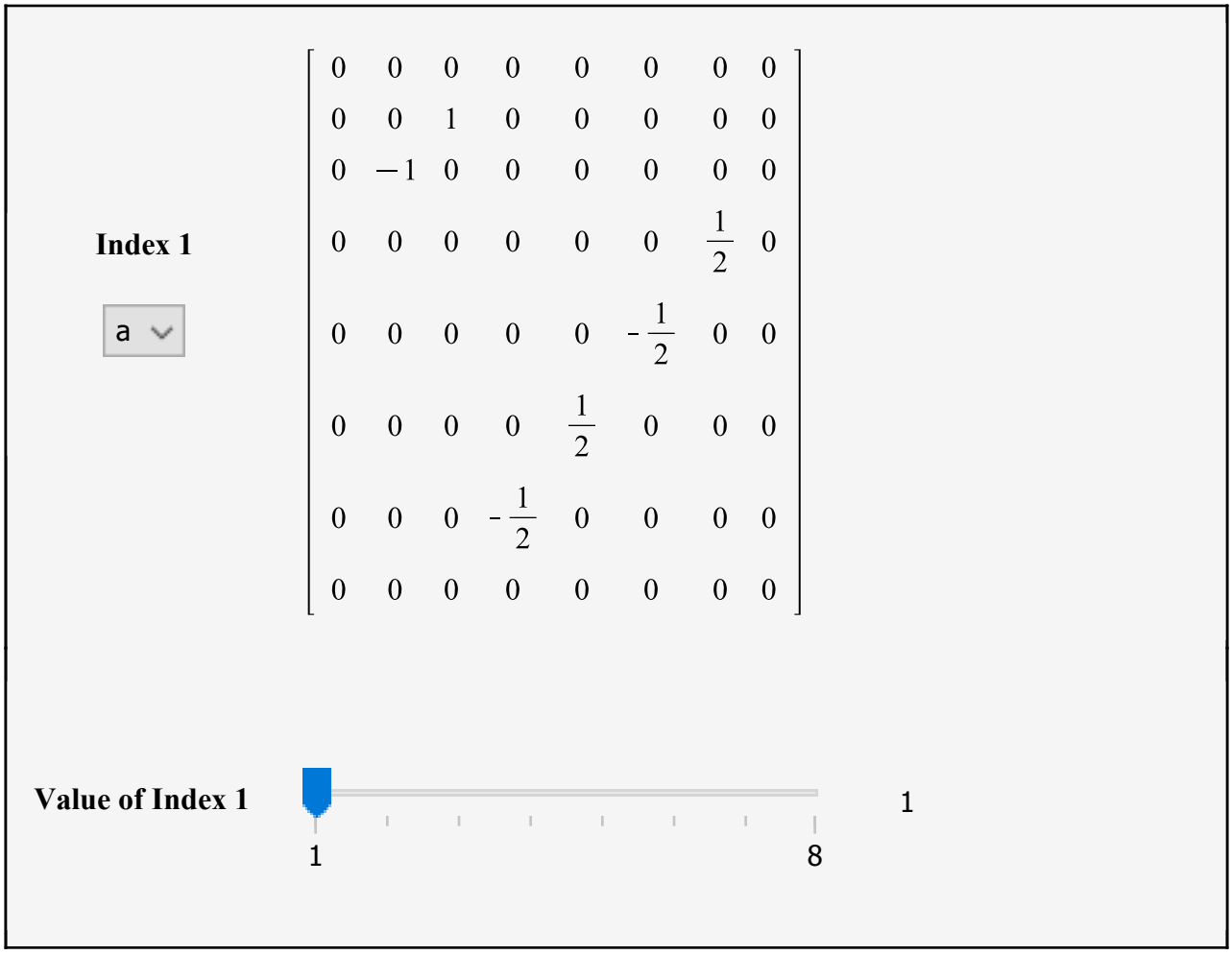
> *FSU3*[1, *b*, *c*, *matrix*]

$$f_{su3_{1,\,b,\,c}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

or, for a more general exploration of the components of $f_{su3_{a,\,b,\,c}}$ you can use the command TensorArray with the option *explore*

> *TensorArray*(*FSU3*[*a*, *b*, *c*], *explore*)

$$f_{su3_{a,\,b,\,c}} \qquad (\mathit{ordering\ of\ free\ indices} = [a, b, c]) \tag{14}$$



The tensorial equation for the Gell-Mann matrices

> (12)

$$\left[\lambda_b, \lambda_c\right]_- = 2\,\mathrm{i}\, f_{su3_{a,\,b,\,c}}\,\lambda_a \qquad (15)$$

is computable for each value of its tensor indices, e.g.

> *SumOverRepeatedIndices*((12))

$$\left[\lambda_b, \lambda_c\right]_- = 2\,\mathrm{i}\left(f_{su3_{1,\,b,\,c}}\,\lambda_1 + f_{su3_{2,\,b,\,c}}\,\lambda_2 + f_{su3_{3,\,b,\,c}}\,\lambda_3 + f_{su3_{4,\,b,\,c}}\,\lambda_4 + f_{su3_{5,\,b,\,c}}\,\lambda_5 + f_{su3_{6,\,b,\,c}}\,\lambda_6 + f_{su3_{7,\,b,\,c}}\,\lambda_7 + f_{su3_{8,\,b,\,c}}\,\lambda_8\right) \qquad (16)$$

> *eval*((16), [b = 4, c = 5])

$$\left[\lambda_4, \lambda_5\right]_- = 2\,\mathrm{i}\left(\frac{\lambda_3}{2} + \frac{\sqrt{3}\,\lambda_8}{2}\right) \qquad (17)$$

Activating the left-hand side,

> *value*(**(17)**)

$$2\,\mathrm{i}\,f_{su3_{4,\,5,\,a}}\,\lambda_a = 2\,\mathrm{i}\left(\frac{\lambda_3}{2}+\frac{\sqrt{3}\,\lambda_8}{2}\right) \tag{18}$$

> *expand*(*SumOverRepeatedIndices*(**(18)**))

$$\mathrm{i}\,\lambda_3+\mathrm{i}\sqrt{3}\,\lambda_8 = \mathrm{i}\,\lambda_3+\mathrm{i}\sqrt{3}\,\lambda_8 \tag{19}$$

To see all the components of **(12)** $\equiv \left[\lambda_b,\lambda_c\right]_- = 2\,\mathrm{I} f_{su3_{a,\,b,\,c}}\,\lambda_a$ at once you can use TensorArray

> *TensorArray*(**(12)**)

$$\begin{aligned}
&\Big[\Big[\left[\lambda_1,\lambda_1\right]_- = 0,\ \left[\lambda_1,\lambda_2\right]_- = 2\,\mathrm{i}\,\lambda_3,\ \left[\lambda_1,\lambda_3\right]_- = -2\,\mathrm{i}\,\lambda_2,\ \left[\lambda_1,\lambda_4\right]_- = \mathrm{i}\,\lambda_7,\ \left[\lambda_1,\lambda_5\right]_- = -\mathrm{i}\,\lambda_6,\\
&\quad \left[\lambda_1,\lambda_6\right]_- = \mathrm{i}\,\lambda_5,\ \left[\lambda_1,\lambda_7\right]_- = -\mathrm{i}\,\lambda_4,\ \left[\lambda_1,\lambda_8\right]_- = 0\Big],\\
&\quad \Big[\left[\lambda_2,\lambda_1\right]_- = -2\,\mathrm{i}\,\lambda_3,\ \left[\lambda_2,\lambda_2\right]_- = 0,\ \left[\lambda_2,\lambda_3\right]_- = 2\,\mathrm{i}\,\lambda_1,\ \left[\lambda_2,\lambda_4\right]_- = \mathrm{i}\,\lambda_6,\ \left[\lambda_2,\lambda_5\right]_- = \mathrm{i}\,\lambda_7,\\
&\quad \left[\lambda_2,\lambda_6\right]_- = -\mathrm{i}\,\lambda_4,\ \left[\lambda_2,\lambda_7\right]_- = -\mathrm{i}\,\lambda_5,\ \left[\lambda_2,\lambda_8\right]_- = 0\Big],\\
&\quad \Big[\left[\lambda_3,\lambda_1\right]_- = 2\,\mathrm{i}\,\lambda_2,\ \left[\lambda_3,\lambda_2\right]_- = -2\,\mathrm{i}\,\lambda_1,\ \left[\lambda_3,\lambda_3\right]_- = 0,\ \left[\lambda_3,\lambda_4\right]_- = \mathrm{i}\,\lambda_5,\ \left[\lambda_3,\lambda_5\right]_- = -\mathrm{i}\,\lambda_4,\\
&\quad \left[\lambda_3,\lambda_6\right]_- = -\mathrm{i}\,\lambda_7,\ \left[\lambda_3,\lambda_7\right]_- = \mathrm{i}\,\lambda_6,\ \left[\lambda_3,\lambda_8\right]_- = 0\Big],\\
&\quad \Big[\left[\lambda_4,\lambda_1\right]_- = -\mathrm{i}\,\lambda_7,\ \left[\lambda_4,\lambda_2\right]_- = -\mathrm{i}\,\lambda_6,\ \left[\lambda_4,\lambda_3\right]_- = -\mathrm{i}\,\lambda_5,\ \left[\lambda_4,\lambda_4\right]_- = 0,\ \left[\lambda_4,\lambda_5\right]_-\\
&\quad = \mathrm{i}\left(\sqrt{3}\,\lambda_8+\lambda_3\right),\ \left[\lambda_4,\lambda_6\right]_- = \mathrm{i}\,\lambda_2,\ \left[\lambda_4,\lambda_7\right]_- = \mathrm{i}\,\lambda_1,\ \left[\lambda_4,\lambda_8\right]_- = -\mathrm{i}\sqrt{3}\,\lambda_5\Big],\\
&\quad \Big[\left[\lambda_5,\lambda_1\right]_- = \mathrm{i}\,\lambda_6,\ \left[\lambda_5,\lambda_2\right]_- = -\mathrm{i}\,\lambda_7,\ \left[\lambda_5,\lambda_3\right]_- = \mathrm{i}\,\lambda_4,\ \left[\lambda_5,\lambda_4\right]_- = -\mathrm{i}\left(\sqrt{3}\,\lambda_8+\lambda_3\right),\\
&\quad \left[\lambda_5,\lambda_5\right]_- = 0,\ \left[\lambda_5,\lambda_6\right]_- = -\mathrm{i}\,\lambda_1,\ \left[\lambda_5,\lambda_7\right]_- = \mathrm{i}\,\lambda_2,\ \left[\lambda_5,\lambda_8\right]_- = \mathrm{i}\sqrt{3}\,\lambda_4\Big],\\
&\quad \Big[\left[\lambda_6,\lambda_1\right]_- = -\mathrm{i}\,\lambda_5,\ \left[\lambda_6,\lambda_2\right]_- = \mathrm{i}\,\lambda_4,\ \left[\lambda_6,\lambda_3\right]_- = \mathrm{i}\,\lambda_7,\ \left[\lambda_6,\lambda_4\right]_- = -\mathrm{i}\,\lambda_2,\ \left[\lambda_6,\lambda_5\right]_- = \mathrm{i}\,\lambda_1,\\
&\quad \left[\lambda_6,\lambda_6\right]_- = 0,\ \left[\lambda_6,\lambda_7\right]_- = \mathrm{i}\left(-\lambda_3+\sqrt{3}\,\lambda_8\right),\ \left[\lambda_6,\lambda_8\right]_- = -\mathrm{i}\sqrt{3}\,\lambda_7\Big],\\
&\quad \Big[\left[\lambda_7,\lambda_1\right]_- = \mathrm{i}\,\lambda_4,\ \left[\lambda_7,\lambda_2\right]_- = \mathrm{i}\,\lambda_5,\ \left[\lambda_7,\lambda_3\right]_- = -\mathrm{i}\,\lambda_6,\ \left[\lambda_7,\lambda_4\right]_- = -\mathrm{i}\,\lambda_1,\ \left[\lambda_7,\lambda_5\right]_- = -\mathrm{i}\,\lambda_2,\\
&\quad \left[\lambda_7,\lambda_6\right]_- = -\mathrm{i}\left(-\lambda_3+\sqrt{3}\,\lambda_8\right),\ \left[\lambda_7,\lambda_7\right]_- = 0,\ \left[\lambda_7,\lambda_8\right]_- = \mathrm{i}\sqrt{3}\,\lambda_6\Big],\\
&\quad \Big[\left[\lambda_8,\lambda_1\right]_- = 0,\ \left[\lambda_8,\lambda_2\right]_- = 0,\ \left[\lambda_8,\lambda_3\right]_- = 0,\ \left[\lambda_8,\lambda_4\right]_- = \mathrm{i}\sqrt{3}\,\lambda_5,\ \left[\lambda_8,\lambda_5\right]_- = -\mathrm{i}\sqrt{3}\,\lambda_4,\\
&\quad \left[\lambda_8,\lambda_6\right]_- = \mathrm{i}\sqrt{3}\,\lambda_7,\ \left[\lambda_8,\lambda_7\right]_- = -\mathrm{i}\sqrt{3}\,\lambda_6,\ \left[\lambda_8,\lambda_8\right]_- = 0\Big]\Big]
\end{aligned} \tag{20}$$

To represent, in what follows, the interaction Lagrangians for QCD and the Electro-Weak sector as sums over leptons and quarks, all of them fermions, it is useful to introduce two anticommutative prefixes to

be used as summation indices

> $Setup(anticommutativeprefix = \{f_L, f_Q\})$

$$[anticommutativeprefix = \{f_L, f_Q\}] \quad (21)$$

> $CompactDisplay(\ (f_L, f_Q)(X)\ )$

$$f_L(X) \text{ will now be displayed as } f_L$$
$$f_Q(X) \text{ will now be displayed as } f_Q \quad (22)$$

The Quantum Chromodynamics (QCD) sector of the Standard Model and its interaction Lagrangian

QCD is about the interaction between quarks and gluons and the self-interaction of the latter. Quarks are implemented as tensors with one spinor and one SU(3) fundamental representation (1..3) indices. Unless set otherwise, according to the starting message these indices are represented by *lowercaselatin_is* and *uppercaselatin_ah* letters. Gluons are tensors with one spacetime and one SU(3) adjoint representation index (1..8), respectively represented by *greek* and *lowercaselatin_ah* letters, and g_s is the QCD coupling constant.

The interaction Lagrangian for the QCD can then be introduced as the sum of two terms

> $L_{QCD} := L_{QG} + L_{GG}$

$$L_{QCD} := L_{QG} + L_{GG} \quad (23)$$

where L_{QG} represents the part involving the interaction between quarks and gluons, and L_{GG} the part related to the self-interaction between gluons. L_{QG} is given by

> $L_{QG} := \frac{g_s}{2} \cdot Dgamma[\mathrm{mu}][k, j] \cdot GluonField[\mathrm{mu}, a](X) \cdot Glambda[a][A, B]$
> $\cdot \%add(conjugate(f_Q[k, A](X)) \cdot f_Q[j, B](X), f_Q = StandardModel:\text{-}Quarks)$

$$L_{QG} := \frac{g_s \sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}, \mathbf{d}, \mathbf{s}, \mathbf{b}]} \overline{f_{Q_{k, A}}} f_{Q_{j, B}} \mathbf{G}_{\mu, a} \left(\lambda_a\right)_{A, B} \left(\gamma^{\mu}\right)_{k, j}}{2} \quad (24)$$

The self-interactions of the gluons L_{GG} can be written using the structure constants $f_{su3_{a, b, c}}$ and the Gell-Mann matrices λ_a

> $L_{GG} := -g_s \cdot FSU3[a, b, c] \cdot \left(d\_[\mathrm{mu}](GluonField[\mathrm{nu}, a](X), [X]) \cdot GluonField[\sim \mathrm{mu}, b](X) \right.$

$\cdot GluonField[\sim \mathrm{nu}, c](X)$

$+ \frac{g_s}{4} \cdot FSU3[e, d, c] \cdot GluonField[\mathrm{mu}, a](X) \cdot GluonField[\mathrm{lambda}, b](X) \cdot GluonField[\sim \mathrm{mu},$

$\left. e](X) \cdot GluonField[\sim \mathrm{lambda}, d](X) \right)$

$$L_{GG} := -g_s f_{su3_{a,b,c}} \left(\partial_\mu \left(\mathbf{G}_{\nu,a} \right) \mathbf{G}^{\mu}{}_{b} \mathbf{G}^{\nu}{}_{c} - \frac{g_s f_{su3_{c,d,e}} \mathbf{G}_{\mu,a} \mathbf{G}_{\lambda,b} \mathbf{G}^{\mu}{}_{e} \mathbf{G}^{\lambda}{}_{d}}{4} \right) \quad (25)$$

From where

> L_{QCD}

$$\frac{g_s \left(\overline{\mathbf{u}_{k,A}} \mathbf{u}_{j,B} + \overline{\mathbf{c}_{k,A}} \mathbf{c}_{j,B} + \overline{\mathbf{t}_{k,A}} \mathbf{t}_{j,B} + \overline{\mathbf{d}_{k,A}} \mathbf{d}_{j,B} + \overline{\mathbf{s}_{k,A}} \mathbf{s}_{j,B} + \overline{\mathbf{b}_{k,A}} \mathbf{b}_{j,B} \right) \mathbf{G}_{\mu,a} \left(\lambda_a \right)_{A,B} \left(\gamma^{\mu} \right)_{k,j}}{2} \quad (26)$$

$$+ g_s f_{su3_{a,b,c}} \left(\partial_\mu \left(\mathbf{G}_{\nu,a} \right) \mathbf{G}^{\mu}{}_{b} \mathbf{G}^{\nu}{}_{c} - \frac{g_s f_{su3_{c,d,e}} \mathbf{G}_{\mu,a} \mathbf{G}_{\lambda,b} \mathbf{G}^{\mu}{}_{e} \mathbf{G}^{\lambda}{}_{d}}{4} \right)$$

> $L_{QCD} := value(L_{QCD})$

$L_{QCD} :=$ (27)

$$\frac{g_s \left(\overline{\mathbf{u}_{k,A}} \mathbf{u}_{j,B} + \overline{\mathbf{c}_{k,A}} \mathbf{c}_{j,B} + \overline{\mathbf{t}_{k,A}} \mathbf{t}_{j,B} + \overline{\mathbf{d}_{k,A}} \mathbf{d}_{j,B} + \overline{\mathbf{s}_{k,A}} \mathbf{s}_{j,B} + \overline{\mathbf{b}_{k,A}} \mathbf{b}_{j,B} \right) \mathbf{G}_{\mu,a} \left(\lambda_a \right)_{A,B} \left(\gamma^{\mu} \right)_{k,j}}{2}$$

$$+ g_s f_{su3_{a,b,c}} \left(\partial_\mu \left(\mathbf{G}_{\nu,a} \right) \mathbf{G}^{\mu}{}_{b} \mathbf{G}^{\nu}{}_{c} - \frac{g_s f_{su3_{c,d,e}} \mathbf{G}_{\mu,a} \mathbf{G}_{\lambda,b} \mathbf{G}^{\mu}{}_{e} \mathbf{G}^{\lambda}{}_{d}}{4} \right)$$

Each of these terms has different contributions to a scattering amplitude. For example, take the first term with the interaction between *Up* quarks and gluons and last one with the self-interaction between four gluons.

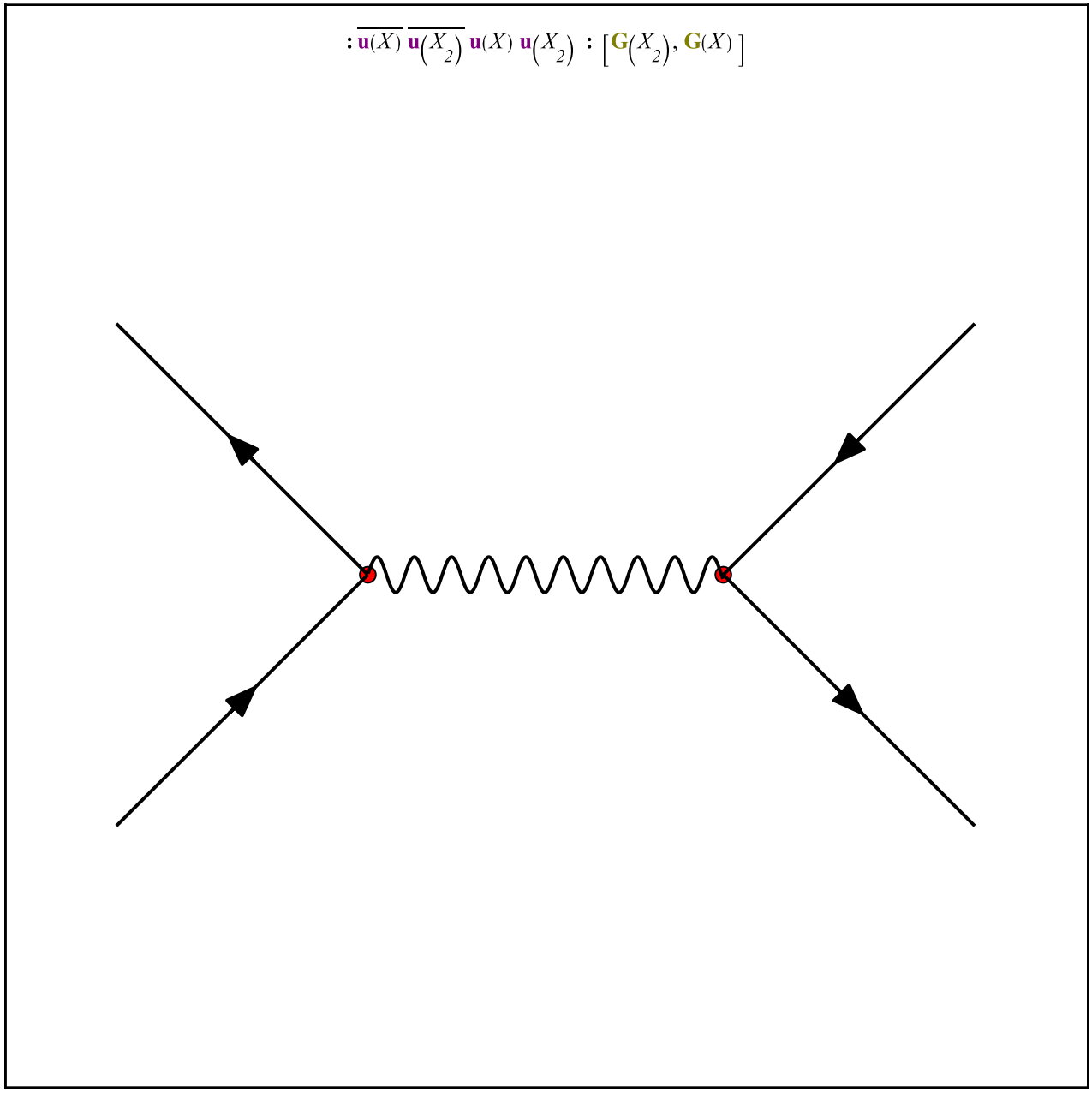
> $L_{uG} := op(1, expand(value(L_{QCD})))$

$$L_{uG} := \frac{g_s \left(\gamma^{\mu} \right)_{k,j} \overline{\mathbf{u}_{k,A}} \mathbf{u}_{j,B} \mathbf{G}_{\mu,a} \left(\lambda_a \right)_{A,B}}{2} \quad (28)$$

The amplitude for the process with two incoming and two outgoing *Up* quarks (particle and antiparticle)

> $FeynmanDiagrams(L_{uG}, incomingparticles = [Up, conjugate(Up)], outgoingparticles = [Up,$

$conjugate(Up)\,],\ numberofloops = 0,\ diagrams\Big)$



$$-\frac{1}{16\,\pi^2\left(\left(P_{1_\kappa}+P_{2_\kappa}\right)\left(P_1{}^\kappa+P_2{}^\kappa\right)+i\,\epsilon\right)}\Big(i\,(\boldsymbol{u}_u)_{l,\,C}\left(\overrightarrow{P_1}\right)\overline{(\boldsymbol{v}_u)_{m,\,E}\left(\overrightarrow{P_2}\right)}\ \overline{(\boldsymbol{u}_u)_{n,\,F}\left(\overrightarrow{P_3}\right)}\,(\boldsymbol{v}_u)_{p,\,G}\left(\overrightarrow{P_4}\right)g_s{}^2\left(\gamma^\alpha\right)_{n,\,p}\left(\gamma^\nu\right)_{m,\,l}g_{\alpha,\,\nu}\,\delta_{b,\,c}\,\delta\Big(-P_3{}^\beta-P_4{}^\beta+P_1{}^\beta+P_2{}^\beta\Big)\left(\lambda_c\right)_{F,\,G}\left(\lambda_b\right)_{E,\,C}\Big)+\frac{1}{16\,\pi^2\left(\left(P_1{}^\kappa-P_3{}^\kappa\right)\left(P_{1_\kappa}-P_{3_\kappa}\right)+i\,\epsilon\right)}\Big(i\,(\boldsymbol{u}_u)_{l,\,C}\Big(\overrightarrow{P_1}\Big)\ \overline{(\boldsymbol{v}_u)_{m,\,E}\left(\overrightarrow{P_2}\right)}\ \overline{(\boldsymbol{u}_u)_{n,\,F}\left(\overrightarrow{P_3}\right)}\,(\boldsymbol{v}_u)_{p,\,G}\left(\overrightarrow{P_4}\right)g_s{}^2\left(\gamma^\alpha\right)_{m,\,p}\left(\gamma^\nu\right)_{n,\,l}g_{\alpha,\,\nu}\,\delta_{b,\,c}\,\delta\Big(-P_3{}^\beta-P_4{}^\beta \tag{29}$$

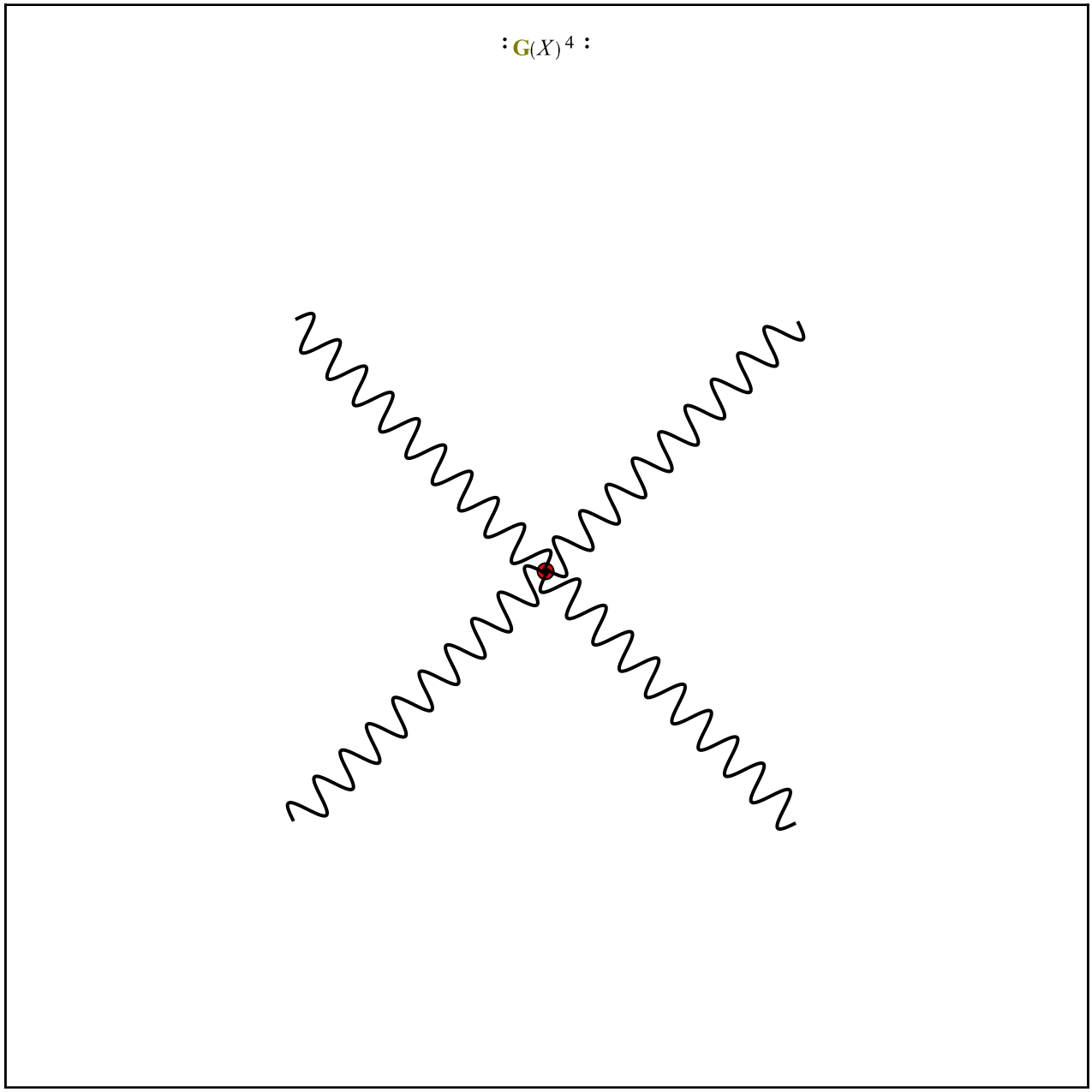
$$+ P_1{}^{\beta} + P_2{}^{\beta} \Big) \left(\lambda_c\right)_{E,\,G} \left(\lambda_b\right)_{F,\,C} \Big)$$

> $L_{GGGG} := op\left(-1, expand\left(L_{QCD}\right)\right)$

$$L_{GGGG} := -\frac{g_s{}^2 f_{su3_{a,\,b,\,c}} f_{su3_{c,\,d,\,e}} \mathbf{G}_{\lambda,\,b} \mathbf{G}_{\mu,\,a} \mathbf{G}^{\lambda}{}_{d} \mathbf{G}^{\mu}{}_{e}}{4} \tag{30}$$

The amplitude for the process with two incoming and two outgoing gluons

> $FeynmanDiagrams\left(L_{GGGG}, incomingparticles = [GluonField, GluonField], outgoingparticles = [GluonField, GluonField], numberofloops = 0, diagrams\right)$



$$\frac{1}{\pi^2 \sqrt{E_1 E_2 E_3 E_4}} \left(-\frac{i}{16} g_s^2 \delta\left(-P_3^{\sigma} - P_4^{\sigma} + P_1^{\sigma} + P_2^{\sigma} \right) \left(\boldsymbol{\varepsilon}_G\right)_{\nu, f}\left(\vec{P_1}\right) \left(\boldsymbol{\varepsilon}_G\right)_{\alpha, g}\left(\vec{P_2}\right) \overline{\left(\boldsymbol{\varepsilon}_G\right)_{\beta, h}\left(\vec{P_3}\right)} \overline{\left(\boldsymbol{\varepsilon}_G\right)_{\kappa, a1}\left(\vec{P_4}\right)} \left(\left(f_{su3_{c,g,h}} f_{su3_{a1,c,f}} - f_{su3_{c,f,g}} f_{su3_{a1,c,h}} \right) g^{\beta,\nu} g^{\alpha,\kappa} + g^{\kappa,\nu} \left(f_{su3_{a1,c,g}} f_{su3_{c,f,h}} + f_{su3_{c,f,g}} f_{su3_{a1,c,h}} \right) g^{\alpha,\beta} + \left(-f_{su3_{c,g,h}} f_{su3_{a1,c,f}} - f_{su3_{a1,c,g}} f_{su3_{c,f,h}} \right) g^{\alpha,\nu} g^{\beta,\kappa} \right) \right) \tag{31}$$

The Electroweak sector of the Standard Model and its interaction Lagrangian

The computation of scattering amplitudes is performed with the model after symmetry breaking. The electro-weak interaction *before symmetry breaking*, from where the formulation *after symmetry breaking* is derived, can be expressed as a sum of four terms mentioned in the Wikipedia page for the weak interaction

> $L_{EW} := L_g + L_f + L_h + L_y$

$$L_{EW} := L_g + L_f + L_h + L_y \tag{32}$$

Out of these four, in the Maple 2022.0 implementation of *StandardModel* it is possible to represent the first term, L_g, the kinetic term for the $\mathbf{W}_{\mu, J}$ and $\mathbf{B}_{\mu}$ vector bosons

> $L_g := -\frac{1}{4} \cdot \left(WFieldStrength[\mu, \nu, J]^2 + BFieldStrength[\mu, \nu]^2 \right)$

$$L_g := -\frac{\mathbb{W}_{\mu,\nu,J} \mathbb{W}^{\mu,\nu}{}_{J}}{4} - \frac{\mathbb{B}_{\mu,\nu} \mathbb{B}^{\mu,\nu}}{4} \tag{33}$$

Introducing the definitions of these tensors we have

> $BFieldStrength[definition], WFieldStrength[definition]$

$$\mathbb{B}_{\mu,\nu} = \partial_{\mu}\left(\mathbf{B}_{\nu}\right) - \partial_{\nu}\left(\mathbf{B}_{\mu}\right), \mathbb{W}_{\mu,\nu,J} = \partial_{\mu}\left(\mathbf{W}_{\nu,J}\right) - \partial_{\nu}\left(\mathbf{W}_{\mu,J}\right) + g_w \epsilon_{J,K,L} \mathbf{W}_{\mu,K} \mathbf{W}_{\nu,L} \tag{34}$$

> $L_g := SubstituteTensor\left((34), L_g \right)$

$$L_g := -\frac{1}{4} \left(\left(\partial_{\mu}\left(\mathbf{W}_{\nu,J}\right) - \partial_{\nu}\left(\mathbf{W}_{\mu,J}\right) + g_w \epsilon_{J,K,L} \mathbf{W}_{\mu,K} \mathbf{W}_{\nu,L} \right) \left(\partial^{\mu}\left(\mathbf{W}^{\nu}{}_{J}\right) - \partial^{\nu}\left(\mathbf{W}^{\mu}{}_{J}\right) + g_w \epsilon_{J,M,N} \mathbf{W}^{\mu}{}_{M} \mathbf{W}^{\nu}{}_{N} \right) \right) - \frac{\left(\partial_{\mu}\left(\mathbf{B}_{\nu}\right) - \partial_{\nu}\left(\mathbf{B}_{\mu}\right) \right) \left(\partial^{\mu}\left(\mathbf{B}^{\nu}\right) - \partial^{\nu}\left(\mathbf{B}^{\mu}\right) \right)}{4} \tag{35}$$

The L_f term is the kinetic term for the fermions of the model *before symmetry breaking*, and their

interaction with the gauge bosons $\mathbf{W}_{\mu,K}$ and $\mathbf{B}_{\mu}$ is through the covariant derivative. Note that the electron field $\mathbf{e}_j$, as well as all the leptons are Dirac spinors that result *after symmetry breaking*. The quarks are also particles that appear through the symmetry breaking mechanism. So the terms you get expanding the covariant derivatives of the leptons and quarks, e.g.

```
> D_[mu](Electron[j](X)) :
  % = expand(%)
```

$$\nabla_{\mu}(\mathbf{e}_j) = \partial_{\mu}(\mathbf{e}_j) + \mathrm{i}\, g_e \mathbf{A}_{\mu}\, \mathbf{e}_j \tag{36}$$

```
> D_[mu](Up[j, A](X)) :
  % = expand(%)
```

$$\nabla_{\mu}(\mathbf{u}_{j,A}) = \partial_{\mu}(\mathbf{u}_{j,A}) - \frac{2\,\mathrm{i}\, g_e \mathbf{A}_{\mu}\, \mathbf{u}_{j,A}}{3} - \frac{\mathrm{i}\, g_s (\lambda_a)_{A,B}\, \mathbf{G}_{\mu,a}\, \mathbf{u}_{j,B}}{2} \tag{37}$$

are of no use for constructing the Lagrangian *before symmetry breaking*. The L_h term involves the Higgs boson *before symmetry breaking* (here too, the HiggsBoson field implemented in the StandardModel in Maple 2022 is the Higgs *after symmetry breaking*) and the L_y formulates the Yukawa interaction with the fermions.

After symmetry breaking

For the purpose of computing scattering amplitudes, the formulation of the interaction Lagrangian *after symmetry breaking* is more relevant; this one is given by

```
> L_EW := L_K + L_N + L_C + L_H + L_HV + L_WWV + L_WWVV + L_Y;
```

$$L_{EW} := L_K + L_N + L_C + L_H + L_{HV} + L_{WWV} + L_{WWVV} + L_{Y;} \tag{38}$$

where we use the notation shown in the Wikipedia page for the weak interaction. As illustration, we compute here the L_K and L_N terms, respectively containing the kinetic terms corresponding to the free fields and the interaction terms between the fermions - leptons and quarks - and the gauge bosons $\mathbf{A}_{\mu}$ and $\mathbf{Z}_{\mu}$.

Following the Wikipedia page mentioned, the kinetic term L_K is given by

```
> L_K := -1/4 ElectromagneticFieldStrength[mu, nu]^2
  - 1/2 WPlusFieldStrength[mu, nu] . WMinusFieldStrength[mu, nu] + 1/2 . m[WField]^2
    . WPlusField[mu] . WMinusField[mu]
  - 1/4 ZFieldStrength[mu, nu]^2 + 1/2 m[ZField]^2 . ZField[mu]^2
```

$$+\frac{1}{2}d\_[\mathrm{mu}](HiggsBoson(X))^2-\frac{m[HiggsBoson]^2}{2}\cdot HiggsBoson(X)^2$$
$$+\%add\left(conjugate\left(f_L[j](X)\right)\cdot\left(Dgamma[\mathrm{mu}][j,k]\cdot i\cdot d\_[\mathrm{mu}]\left(f_L[k](X)\right)-m[f_L]\cdot f_L[j](X)\right),f_L=StandardModel{:}\text{-}Leptons[1..3]\right)$$
$$+\%add\left(conjugate\left(f_L[j](X)\right)\cdot\left(Dgamma[\mathrm{mu}][j,k]\cdot i\cdot d\_[\mathrm{mu}]\left(f_L[k](X)\right)\right),f_L=StandardModel{:}\text{-}Leptons[4..6]\right)$$
$$+\%add\left(conjugate\left(f_Q[j,A](X)\right)\cdot\left(Dgamma[\mathrm{mu}][j,k]\cdot i\cdot d\_[\mathrm{mu}]\left(f_Q[k,A](X)\right)-m[f_Q]\cdot f_Q[j,A](X)\right),f_Q=StandardModel{:}\text{-}Quarks\right)$$

$$L_K := -\frac{\mathbb{F}_{\mu,\nu}\,\mathbb{F}^{\mu,\nu}}{4}-\frac{\mathbb{W}^{+}_{\mu,\nu}\,\mathbb{W}^{-\,\mu,\nu}}{2}+\frac{m_W^2\,W^{+}_{\mu}\,W^{-\,\mu}}{2}-\frac{\mathbb{Z}_{\mu,\nu}\,\mathbb{Z}^{\mu,\nu}}{4}+\frac{m_Z^2\,Z_{\mu}\,Z^{\mu}}{2}+\frac{\partial_{\mu}(\Phi)\,\partial^{\mu}(\Phi)}{2}-\frac{m_\Phi^2\,\Phi^2}{2}+\sum_{f_L=[\mathbf{e},\mu,\tau]}\overline{f_{L_j}}\left(i\,\partial_{\mu}\left(f_{L_k}\right)\left(\gamma^{\mu}\right)_{j,k}-m_{f_L}f_{L_j}\right)+\sum_{f_L=[\nu^{(e)},\nu^{(\mu)},\nu^{(\tau)}]}i\,\overline{f_{L_j}}\,\partial_{\mu}\left(f_{L_k}\right)\left(\gamma^{\mu}\right)_{j,k}+\sum_{f_Q=[\mathbf{u},\mathbf{c},\mathbf{t},\mathbf{d},\mathbf{s},\mathbf{b}]}\overline{f_{Q_{j,A}}}\left(i\,\partial_{\mu}\left(f_{Q_{k,A}}\right)\left(\gamma^{\mu}\right)_{j,k}-m_{f_Q}f_{Q_{j,A}}\right) \quad (39)$$

The inert sums over the leptons and quarks can be activated using value

> value((39))

$$-\frac{\mathbb{F}_{\mu,\nu}\,\mathbb{F}^{\mu,\nu}}{4}-\frac{\mathbb{W}^{+}_{\mu,\nu}\,\mathbb{W}^{-\,\mu,\nu}}{2}+\frac{m_W^2\,W^{+}_{\mu}\,W^{-\,\mu}}{2}-\frac{\mathbb{Z}_{\mu,\nu}\,\mathbb{Z}^{\mu,\nu}}{4}+\frac{m_Z^2\,Z_{\mu}\,Z^{\mu}}{2}+\frac{\partial_{\mu}(\Phi)\,\partial^{\mu}(\Phi)}{2}-\frac{m_\Phi^2\,\Phi^2}{2}+\overline{\mathbf{e}_j}\left(i\,\partial_{\mu}\left(\mathbf{e}_k\right)\left(\gamma^{\mu}\right)_{j,k}-m_{\mathbf{e}}\,\mathbf{e}_j\right)+\overline{\boldsymbol{\mu}_j}\left(i\,\partial_{\mu}\left(\boldsymbol{\mu}_k\right)\left(\gamma^{\mu}\right)_{j,k}-m_{\boldsymbol{\mu}}\,\boldsymbol{\mu}_j\right)+\overline{\boldsymbol{\tau}_j}\left(i\,\partial_{\mu}\left(\boldsymbol{\tau}_k\right)\left(\gamma^{\mu}\right)_{j,k}-m_{\boldsymbol{\tau}}\,\boldsymbol{\tau}_j\right)+i\,\overline{\nu^{(\tau)}{}_j}\,\partial_{\mu}\left(\nu^{(\tau)}{}_k\right)\left(\gamma^{\mu}\right)_{j,k}+i\,\overline{\nu^{(e)}{}_j}\,\partial_{\mu}\left(\nu^{(e)}{}_k\right)\left(\gamma^{\mu}\right)_{j,k}+i\,\overline{\nu^{(\mu)}{}_j}\,\partial_{\mu}\left(\nu^{(\mu)}{}_k\right)\left(\gamma^{\mu}\right)_{j,k}+\overline{\mathbf{u}_{j,A}}\left(i\,\partial_{\mu}\left(\mathbf{u}_{k,A}\right)\left(\gamma^{\mu}\right)_{j,k}-m_{\mathbf{u}}\,\mathbf{u}_{j,A}\right)+\overline{\mathbf{c}_{j,A}}\left(i\,\partial_{\mu}\left(\mathbf{c}_{k,A}\right)\left(\gamma^{\mu}\right)_{j,k}-m_{\mathbf{c}}\,\mathbf{c}_{j,A}\right)+\overline{\mathbf{t}_{j,A}}\left(i\,\partial_{\mu}\left(\mathbf{t}_{k,A}\right)\left(\gamma^{\mu}\right)_{j,k}-m_{\mathbf{t}}\,\mathbf{t}_{j,A}\right)+\overline{\mathbf{d}_{j,A}}\left(i\,\partial_{\mu}\left(\mathbf{d}_{k,A}\right)\left(\gamma^{\mu}\right)_{j,k}-m_{\mathbf{d}}\,\mathbf{d}_{j,A}\right)+\overline{\mathbf{s}_{j,A}}\left(i\,\partial_{\mu}\left(\mathbf{s}_{k,A}\right)\left(\gamma^{\mu}\right)_{j,k}-m_{\mathbf{s}}\,\mathbf{s}_{j,A}\right)+\overline{\mathbf{b}_{j,A}}\left(i\,\partial_{\mu}\left(\mathbf{b}_{k,A}\right)\left(\gamma^{\mu}\right)_{j,k}-m_{\mathbf{b}}\,\mathbf{b}_{j,A}\right) \quad (40)$$

Introducing the definition of the field strengths $\mathbb{F}_{\mu,\nu}$, $\mathbb{W}^{+}_{\mu,\nu}$, $\mathbb{W}^{-}_{\mu,\nu}$ and $\mathbb{Z}_{\mu,\nu}$

> *ElectromagneticFieldStrength*[*definition*]

$$\mathbb{F}_{\mu,\nu} = \partial_{\mu}(\mathbf{A}_{\nu}) - \partial_{\nu}(\mathbf{A}_{\mu}) \tag{41}$$

> *WPlusFieldStrength*[*definition*]

$$\mathbb{W}^{+}_{\mu,\nu} = \partial_{\mu}(\mathbf{W}^{+}_{\nu}) - \partial_{\nu}(\mathbf{W}^{+}_{\mu}) \tag{42}$$

> *WMinusFieldStrength*[*definition*]

$$\mathbb{W}^{-}_{\mu,\nu} = \partial_{\mu}(\mathbf{W}^{-}_{\nu}) - \partial_{\nu}(\mathbf{W}^{-}_{\mu}) \tag{43}$$

> *ZFieldStrength*[*definition*]

$$\mathbb{Z}_{\mu,\nu} = \partial_{\mu}(\mathbf{Z}_{\nu}) - \partial_{\nu}(\mathbf{Z}_{\mu}) \tag{44}$$

> L_K := *SubstituteTensor*(**(41)**, **(42)**, **(43)**, **(44)**, **(40)**)

$$\begin{aligned} L_K := & -\frac{\left(\partial_{\mu}(\mathbf{A}_{\nu}) - \partial_{\nu}(\mathbf{A}_{\mu})\right)\left(\partial^{\mu}(\mathbf{A}^{\nu}) - \partial^{\nu}(\mathbf{A}^{\mu})\right)}{4} \\ & - \frac{\left(\partial_{\mu}(\mathbf{W}^{+}_{\nu}) - \partial_{\nu}(\mathbf{W}^{+}_{\mu})\right)\left(\partial^{\mu}(\mathbf{W}^{-\nu}) - \partial^{\nu}(\mathbf{W}^{-\mu})\right)}{2} + \frac{m_{\mathbf{W}}^{2}\,\mathbf{W}^{+}_{\mu}\,\mathbf{W}^{-\mu}}{2} \\ & - \frac{\left(\partial_{\mu}(\mathbf{Z}_{\nu}) - \partial_{\nu}(\mathbf{Z}_{\mu})\right)\left(\partial^{\mu}(\mathbf{Z}^{\nu}) - \partial^{\nu}(\mathbf{Z}^{\mu})\right)}{4} + \frac{m_{\mathbf{Z}}^{2}\,\mathbf{Z}_{\mu}\,\mathbf{Z}^{\mu}}{2} + \frac{\partial_{\mu}(\Phi)\,\partial^{\mu}(\Phi)}{2} \\ & - \frac{m_{\Phi}^{2}\,\Phi^{2}}{2} + \overline{\mathbf{e}}_{j}\left(\mathrm{i}\,\partial_{\mu}(\mathbf{e}_{k})\left(\gamma^{\mu}\right)_{j,k} - m_{\mathbf{e}}\,\mathbf{e}_{j}\right) + \overline{\boldsymbol{\mu}}_{j}\left(\mathrm{i}\,\partial_{\mu}(\boldsymbol{\mu}_{k})\left(\gamma^{\mu}\right)_{j,k} - m_{\boldsymbol{\mu}}\,\boldsymbol{\mu}_{j}\right) + \\ & \overline{\boldsymbol{\tau}}_{j}\left(\mathrm{i}\,\partial_{\mu}(\boldsymbol{\tau}_{k})\left(\gamma^{\mu}\right)_{j,k} - m_{\boldsymbol{\tau}}\,\boldsymbol{\tau}_{j}\right) + \mathrm{i}\,\overline{\boldsymbol{\nu}^{(\tau)}}_{j}\,\partial_{\mu}\left(\boldsymbol{\nu}^{(\tau)}{}_{k}\right)\left(\gamma^{\mu}\right)_{j,k} + \mathrm{i}\,\overline{\boldsymbol{\nu}^{(\mathrm{e})}}_{j}\,\partial_{\mu}\left(\boldsymbol{\nu}^{(\mathrm{e})}{}_{k}\right)\left(\gamma^{\mu}\right)_{j,k} + \mathrm{i} \\ & \overline{\boldsymbol{\nu}^{(\mu)}}_{j}\,\partial_{\mu}\left(\boldsymbol{\nu}^{(\mu)}{}_{k}\right)\left(\gamma^{\mu}\right)_{j,k} + \overline{\mathbf{u}_{j,A}}\left(\mathrm{i}\,\partial_{\mu}(\mathbf{u}_{k,A})\left(\gamma^{\mu}\right)_{j,k} - m_{\mathbf{u}}\,\mathbf{u}_{j,A}\right) + \overline{\mathbf{c}_{j,A}}\left(\mathrm{i}\,\partial_{\mu}(\mathbf{c}_{k,A})\left(\gamma^{\mu}\right)_{j,k}\right. \\ & \left. - m_{\mathbf{c}}\,\mathbf{c}_{j,A}\right) + \overline{\mathbf{t}_{j,A}}\left(\mathrm{i}\,\partial_{\mu}(\mathbf{t}_{k,A})\left(\gamma^{\mu}\right)_{j,k} - m_{\mathbf{t}}\,\mathbf{t}_{j,A}\right) + \overline{\mathbf{d}_{j,A}}\left(\mathrm{i}\,\partial_{\mu}(\mathbf{d}_{k,A})\left(\gamma^{\mu}\right)_{j,k} - m_{\mathbf{d}}\,\mathbf{d}_{j,A}\right) + \\ & \overline{\mathbf{s}_{j,A}}\left(\mathrm{i}\,\partial_{\mu}(\mathbf{s}_{k,A})\left(\gamma^{\mu}\right)_{j,k} - m_{\mathbf{s}}\,\mathbf{s}_{j,A}\right) + \overline{\mathbf{b}_{j,A}}\left(\mathrm{i}\,\partial_{\mu}(\mathbf{b}_{k,A})\left(\gamma^{\mu}\right)_{j,k} - m_{\mathbf{b}}\,\mathbf{b}_{j,A}\right) \end{aligned} \tag{45}$$

The neutral current Lagrangian containing the interactions between fermions and the gauge bosons $\mathbf{A}_{\mu}$ and $\mathbf{Z}_{\mu}$ is expressed in terms of the electromagnetic and weak currents $J_{E,\mu}$ and $J_{W,\mu}$ as

> $L_N := g_e \cdot J[E,\mu] \cdot ElectromagneticField[\mu](X) + \frac{g_w}{\cos(WeinbergAngle)} \cdot (J[W, \mathrm{mu}] - \sin(WeinbergAngle)^2 \cdot J[E, \mathrm{mu}]) \cdot ZField[\mathrm{mu}](X)$

$$L_N := g_e\,J_{E,\mu}\,\mathbf{A}_{\mu} + \frac{g_w\left(J_{W,\mu} - \sin(\boldsymbol{\theta}_{\mathbf{w}})^2\,J_{E,\mu}\right)\mathbf{Z}_{\mu}}{\cos(\boldsymbol{\theta}_{\mathbf{w}})} \tag{46}$$

In turn, these currents are expressed as

```
> J[E, μ] := %q_e·Dgamma[mu][k, j]·%add(conjugate(f_L[k](X))·f_L[j](X), f_L = [Electron, Muon, Tauon])
  + %q_u·Dgamma[mu][k, j]·%add(conjugate(f_Q[k, A](X))·f_Q[j, A](X), f_Q = [Up, Charm, Top])
  + %q_d·Dgamma[mu][k, j]·%add(conjugate(f_Q[k, A](X))·f_Q[j, A](X), f_Q = [Down, Strange, Bottom])
```

$$J_{E,\mu} := q_e \left(\gamma_\mu\right)_{k,j} \sum_{f_L = [\mathbf{e}, \mu, \tau]} \overline{f_{L_k}} f_{L_j} + q_u \left(\gamma_\mu\right)_{k,j} \sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \overline{f_{Q_{k,A}}} f_{Q_{j,A}} + q_d \left(\gamma_\mu\right)_{k,j} \sum_{f_Q = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \overline{f_{Q_{k,A}}} f_{Q_{j,A}} \tag{47}$$

To activate only the sum over the different kinds of fermions,

```
> J[E, mu] := eval((47), %add = add)
```

$$J_{E,\mu} := q_e \left(\gamma_\mu\right)_{k,j} \left(\overline{\mathbf{e}_k} \mathbf{e}_j + \overline{\mu_k} \mu_j + \overline{\tau_k} \tau_j\right) + q_u \left(\gamma_\mu\right)_{k,j} \left(\overline{\mathbf{u}_{k,A}} \mathbf{u}_{j,A} + \overline{\mathbf{c}_{k,A}} \mathbf{c}_{j,A} + \overline{\mathbf{t}_{k,A}} \mathbf{t}_{j,A}\right) + q_d \left(\gamma_\mu\right)_{k,j} \left(\overline{\mathbf{d}_{k,A}} \mathbf{d}_{j,A} + \overline{\mathbf{s}_{k,A}} \mathbf{s}_{j,A} + \overline{\mathbf{b}_{k,A}} \mathbf{b}_{j,A}\right) \tag{48}$$

To activate the sums and also the inert representations of the different charges you can use the value command

```
> J[E, mu] := value((47))
```

$$J_{E,\mu} := -\left(\gamma_\mu\right)_{k,j} \left(\overline{\mathbf{e}_k} \mathbf{e}_j + \overline{\mu_k} \mu_j + \overline{\tau_k} \tau_j\right) + \frac{2 \left(\gamma_\mu\right)_{k,j} \left(\overline{\mathbf{u}_{k,A}} \mathbf{u}_{j,A} + \overline{\mathbf{c}_{k,A}} \mathbf{c}_{j,A} + \overline{\mathbf{t}_{k,A}} \mathbf{t}_{j,A}\right)}{3} - \frac{\left(\gamma_\mu\right)_{k,j} \left(\overline{\mathbf{d}_{k,A}} \mathbf{d}_{j,A} + \overline{\mathbf{s}_{k,A}} \mathbf{s}_{j,A} + \overline{\mathbf{b}_{k,A}} \mathbf{b}_{j,A}\right)}{3} \tag{49}$$

For the weak current, from the Wikipedia reference mentioned,

```
> J[W, μ] := Dgamma[mu][k, j]·(KroneckerDelta[j, l] − Dgamma[5][j, l])·(
  %I_e·%add(conjugate(f_L[k](X))·f_L[l](X), f_L = StandardModel:-Leptons[1..3])
  + %I_n·%add(conjugate(f_L[k](X))·f_L[l](X), f_L = StandardModel:-Leptons[4..6])
  + %I_u·%add(conjugate(f_Q[k, A](X))·f_Q[l, A](X), f_Q = StandardModel:-Quarks[1..3])
  + %I_d·%add(conjugate(f_Q[k, A](X))·f_Q[l, A](X), f_Q = StandardModel:-Quarks[4..6]) )
```

$$J_{W,\mu} := \left(\gamma_\mu\right)_{k,j} \left(\delta_{j,l} - \left(\gamma_5\right)_{j,l}\right) \left(I_e \sum_{f_L = [\mathbf{e}, \mu, \tau]} \overline{f_{L_k}} f_{L_l} + I_n \sum_{f_L = [\nu_{(e)}, \nu_{(\mu)}, \nu_{(\tau)}]} \overline{f_{L_k}} f_{L_l} + I_u \sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \right. \tag{50}$$

$$\overline{f_{Q_{k,A}}} f_{Q_{l,A}} + I_d \sum_{f_Q = [\mathbf{d},\mathbf{s},\mathbf{b}]} \overline{f_{Q_{k,A}}} f_{Q_{l,A}} \Bigg)$$

To activate only the sums,

> $J[W,\mu] := eval(\,(50),\, \%add = add)$

$$J_{W,\mu} := (\gamma_\mu)_{k,j} \left(\delta_{j,l} - (\gamma_5)_{j,l}\right) \Big(I_e \left(\overline{\mathbf{e}}_k \mathbf{e}_l + \overline{\boldsymbol{\mu}}_k \boldsymbol{\mu}_l + \overline{\boldsymbol{\tau}}_k \boldsymbol{\tau}_l\right) + I_n \left(\overline{\boldsymbol{\nu}^{(e)}}_k \boldsymbol{\nu}^{(e)}{}_l + \overline{\boldsymbol{\nu}^{(\mu)}}_k \boldsymbol{\nu}^{(\mu)}{}_l + \overline{\boldsymbol{\nu}^{(\tau)}}_k \boldsymbol{\nu}^{(\tau)}{}_l\right) + I_u \left(\overline{\mathbf{u}_{k,A}} \mathbf{u}_{l,A} + \overline{\mathbf{c}_{k,A}} \mathbf{c}_{l,A} + \overline{\mathbf{t}_{k,A}} \mathbf{t}_{l,A}\right) + I_d \left(\overline{\mathbf{d}_{k,A}} \mathbf{d}_{l,A} + \overline{\mathbf{s}_{k,A}} \mathbf{s}_{l,A} + \overline{\mathbf{b}_{k,A}} \mathbf{b}_{l,A}\right) \Big) \tag{51}$$

To activate the sums and also the inert representations of the corresponding different isospins I_e, I_n, I_u, I_d you can use the value command

> $J[W, \mathrm{mu}] := value(\,(50)\,)$

$$J_{W,\mu} := (\gamma_\mu)_{k,j} \left(\delta_{j,l} - (\gamma_5)_{j,l}\right) \left(-\frac{\overline{\mathbf{e}}_k \mathbf{e}_l}{2} - \frac{\overline{\boldsymbol{\mu}}_k \boldsymbol{\mu}_l}{2} - \frac{\overline{\boldsymbol{\tau}}_k \boldsymbol{\tau}_l}{2} + \frac{\overline{\boldsymbol{\nu}^{(e)}}_k \boldsymbol{\nu}^{(e)}{}_l}{2} + \frac{\overline{\boldsymbol{\nu}^{(\mu)}}_k \boldsymbol{\nu}^{(\mu)}{}_l}{2} + \frac{\overline{\boldsymbol{\nu}^{(\tau)}}_k \boldsymbol{\nu}^{(\tau)}{}_l}{2} + \frac{\overline{\mathbf{u}_{k,A}} \mathbf{u}_{l,A}}{2} + \frac{\overline{\mathbf{c}_{k,A}} \mathbf{c}_{l,A}}{2} + \frac{\overline{\mathbf{t}_{k,A}} \mathbf{t}_{l,A}}{2} - \frac{\overline{\mathbf{d}_{k,A}} \mathbf{d}_{l,A}}{2} - \frac{\overline{\mathbf{s}_{k,A}} \mathbf{s}_{l,A}}{2} - \frac{\overline{\mathbf{b}_{k,A}} \mathbf{b}_{l,A}}{2} \right) \tag{52}$$

L_N defined in **(46)** in terms of $J_{E,\mu}$ and $J_{W,\mu}$ is then given by

> L_N

$$g_e \Bigg(\left(-\overline{\mathbf{e}}_k \mathbf{e}_j - \overline{\boldsymbol{\mu}}_k \boldsymbol{\mu}_j - \overline{\boldsymbol{\tau}}_k \boldsymbol{\tau}_j\right) (\gamma_\mu)_{k,j} + \left(\frac{2\,\overline{\mathbf{u}_{k,A}} \mathbf{u}_{j,A}}{3} + \frac{2\,\overline{\mathbf{c}_{k,A}} \mathbf{c}_{j,A}}{3} + \frac{2\,\overline{\mathbf{t}_{k,A}} \mathbf{t}_{j,A}}{3} \right) (\gamma_\mu)_{k,j} + \left(-\frac{\overline{\mathbf{d}_{k,A}} \mathbf{d}_{j,A}}{3} - \frac{\overline{\mathbf{s}_{k,A}} \mathbf{s}_{j,A}}{3} - \frac{\overline{\mathbf{b}_{k,A}} \mathbf{b}_{j,A}}{3} \right) (\gamma_\mu)_{k,j} \Bigg) \mathbf{A}^{\mu} + \frac{1}{\cos(\boldsymbol{\theta}_w)} \Bigg(g_w \Bigg((\gamma_\mu)_{k,j} \left(\delta_{j,l} - (\gamma_5)_{j,l}\right) \Bigg(-\frac{\overline{\mathbf{e}}_k \mathbf{e}_l}{2} - \frac{\overline{\boldsymbol{\mu}}_k \boldsymbol{\mu}_l}{2} - \frac{\overline{\boldsymbol{\tau}}_k \boldsymbol{\tau}_l}{2} + \frac{\overline{\boldsymbol{\nu}^{(e)}}_k \boldsymbol{\nu}^{(e)}{}_l}{2} + \frac{\overline{\boldsymbol{\nu}^{(\mu)}}_k \boldsymbol{\nu}^{(\mu)}{}_l}{2} + \frac{\overline{\boldsymbol{\nu}^{(\tau)}}_k \boldsymbol{\nu}^{(\tau)}{}_l}{2} + \frac{\overline{\mathbf{u}_{k,A}} \mathbf{u}_{l,A}}{2} + \frac{\overline{\mathbf{c}_{k,A}} \mathbf{c}_{l,A}}{2} + \frac{\overline{\mathbf{t}_{k,A}} \mathbf{t}_{l,A}}{2} - \frac{\overline{\mathbf{d}_{k,A}} \mathbf{d}_{l,A}}{2} - \frac{\overline{\mathbf{s}_{k,A}} \mathbf{s}_{l,A}}{2} - \frac{\overline{\mathbf{b}_{k,A}} \mathbf{b}_{l,A}}{2} \Bigg) - \sin(\boldsymbol{\theta}_w)^2 \Bigg(\left(-\overline{\mathbf{e}}_k \mathbf{e}_j - \overline{\boldsymbol{\mu}}_k \boldsymbol{\mu}_j - \overline{\boldsymbol{\tau}}_k \boldsymbol{\tau}_j\right) (\gamma_\mu)_{k,j} + \left(\frac{2\,\overline{\mathbf{u}_{k,A}} \mathbf{u}_{j,A}}{3} + \frac{2\,\overline{\mathbf{c}_{k,A}} \mathbf{c}_{j,A}}{3} + \frac{2\,\overline{\mathbf{t}_{k,A}} \mathbf{t}_{j,A}}{3} \right) (\gamma_\mu)_{k,j} + \left(-\frac{\overline{\mathbf{d}_{k,A}} \mathbf{d}_{j,A}}{3} - \frac{\overline{\mathbf{s}_{k,A}} \mathbf{s}_{j,A}}{3} - \frac{\overline{\mathbf{b}_{k,A}} \mathbf{b}_{j,A}}{3} \right) (\gamma_\mu)_{k,j} \Bigg) \Bigg) \mathbf{Z}^{\mu} \Bigg) \tag{53}$$

The structure of indices of this term can be scanned using Check

> $Check(L_N, all)$

The products in the given expression check ok.

The repeated indices per term are: $[\{\ldots\}, \{\ldots\}, \ldots]$, *the free indices are:* $\{\ldots\}$

$$[\{A, j, k, \mu\}, \{A, j, k, l, \mu\}], \varnothing \quad \textbf{(54)}$$

Feynman Diagrams

Feynman Diagrams are the cornerstone of calculations in particle physics (collisions involving from the proton to the Higgs boson), for example at the CERN. As an introduction for people not working in the area, see "Why Feynman Diagrams are so important". In connection, Maple 2020 presented a **full rewriting** of the *FeynmanDiagrams* command including a myriad of new capabilities. Then Maple 2020 included a large number of new options in FeynmanDiagrams, as well as a new FeynmanIntegral module.

In addition, in Maple 2022,

- The most time consuming parts of *FeynmanDiagrams* were revised and rewritten, resulting in a 10x to 100x speed-up depending on the interaction Lagrangian and the number of loops. Also, *FeynmanDiagrams*, that in Maple 2021 can draw diagrams with up to 3 vertices, in Maple 2022 can also draw diagrams with 4 and 5 vertices.
- The *FeynmanIntegral* package to evaluate the Feynman integrals - e.g. those that appear in the output of the *FeynmanDiagrams* command - includes **9 new commands**, covering most of the intermediate or advanced steps involved in that evaluation.

These developments were presented in Computer algebra and Particle Physics - CAPP 2021.

Dramatic speedup

- Among the most significant things in Physics for Maple 2022 is the revision of the code optimizing it for performance, resulting in speedups across the board. For example, Maple 2022 performs the computations of scattering amplitudes 10 to 1000 times faster depending on the case. As an example, consider the interaction Lagrangian for Quantum Chromodynamics (QCD), part of the Standard Model. In QCD, the gluons, that we label here as Gl, are represented by a massless field mediating the force (is the analogous of the photon A in QED), and the model includes 6 different types of quarks: up, down, strange, charm, top and bottom.
- Without making direct use of the StandardModel package, for comparison purposes you can formulate the QCD Lagrangian **(27)** as in Maple 2021 by setting the field operators

```
> Setup(quantumoperators = {Qb, Qt, Qu, Qd, Qc, Qs, Gl}, anticommutativeprefix = {Qb, Qt, Qu,
    Qd, Qc, Qs}, masslessfields = Gl)
```

$$[anticommutativeprefix = \{Qb, Qc, Qd, Qs, Qt, Qu\}, masslessfields = \{Gl, \mathbf{G}, \boldsymbol{\nu}^{(\mu)}, \boldsymbol{\nu}^{(\tau)}, \mathbf{A}, \boldsymbol{\nu}^{(e)}\}, \quad \textbf{(55)}$$

$$quantumoperators = \{\mathbf{B}, \mathbf{b}, \mathbf{c}, \mathbf{d}, Gl, \boldsymbol{\mu}, Qb, Qc, Qd, Qs, Qt, Qu, \mathbf{s}, \boldsymbol{\tau}, \mathbf{t}, \mathbf{u}, \mathbf{W}, \mathbf{Z}, \mathbb{B}, \mathbf{e}, \mathbf{G}, \boldsymbol{\Phi},$$

$\nu^{(\mu)}, \nu^{(\tau)}, \mathbb{W}, \mathbf{W}^{-}, \mathbf{W}^{+}, \mathbb{Z}, \mathbf{A}, \nu^{(e)}, \mathbb{G}, \mathbb{W}^{-}, \mathbb{W}^{+}, \mathbb{F}\}]$

Define as tensors the following fields entering the model

> *Define*(*Gl*[mu, *a*], *Qb*[*k*, *B*], *Qt*[*k*, *B*], *T*[~*a*, *B*, *C*], *f*[~*a*,~*b*,~*c*])

Defined objects with tensor properties

$$\{\mathbf{B}_{\mu}, \mathbf{b}_{a,B}, \mathbf{c}_{a,B}, \gamma_{\mu}, \mathbf{d}_{a,B}, f_{su3}{}_{a,b,c}, Gl_{\mu,a}, \lambda_{a}, \boldsymbol{\mu}_{a}, \sigma_{\mu}, Qb_{k,B}, Qt_{k,B}, \mathbf{s}_{a,B}, T^{a}{}_{B,C}, \boldsymbol{\tau}_{a}, \mathbf{t}_{a,B}, \mathbf{u}_{a,B}, \mathbf{W}_{\mu,K}, \mathbf{Z}_{\mu}, \partial_{\mu}, f^{a,b,c}, f_{L_k}, f_{Q_{k,A}}, g_{\mu,\nu}, \mathbb{B}_{\mu,\nu}, \mathbf{e}_{a}, \mathbf{G}_{\mu,b}, \epsilon_{\alpha,\beta,\mu,\nu}, \nu^{(\mu)}{}_{a}, \nu^{(\tau)}{}_{a}, \mathbb{W}_{\mu,\nu,L}, \mathbf{W}^{-}{}_{\mu}, \mathbf{W}^{+}{}_{\mu}, \mathbb{Z}_{\mu,\nu}, \mathbf{A}_{\mu}, \nu^{(e)}{}_{a}, \mathbb{G}_{\mu,\nu,c}, \mathbb{W}^{-}{}_{\mu,\nu}, \mathbb{W}^{+}{}_{\mu,\nu}, \mathbb{F}_{\mu,\nu}, X_{\mu}\} \quad (56)$$

There are terms for the 6 different types of quarks: *up, down, strange, charm, top* and *bottom*

> $L1 := g \cdot conjugate(Qu[k,B](X))\ Dgamma[\mu][k,j]\ Qu[j,C](X)\ Gl[\mu,a](X)\ T[\sim a,B,C]$

$$L1 := g\, T^{a}{}_{B,C}\, \overline{Qu_{k,B}(X)}\, Qu_{j,C}(X)\, Gl_{\mu,a}(X)\, \left(\gamma^{\mu}\right)_{k,j} \quad (57)$$

> $L2 := g \cdot conjugate(Qd[k,B](X))\ Dgamma[\mu][k,j]\ Qd[j,C](X)\ Gl[\mu,a](X)\ T[\sim a,B,C]$

$$L2 := g\, T^{a}{}_{B,C}\, \overline{Qd_{k,B}(X)}\, Qd_{j,C}(X)\, Gl_{\mu,a}(X)\, \left(\gamma^{\mu}\right)_{k,j} \quad (58)$$

> $L3 := g \cdot conjugate(Qs[k,B](X))\ Dgamma[\mu][k,j]\ Qs[j,C](X)\ Gl[\mu,a](X)\ T[\sim a,B,C]$

$$L3 := g\, T^{a}{}_{B,C}\, \overline{Qs_{k,B}(X)}\, Qs_{j,C}(X)\, Gl_{\mu,a}(X)\, \left(\gamma^{\mu}\right)_{k,j} \quad (59)$$

> $L4 := g \cdot conjugate(Qc[k,B](X))\ Dgamma[\mu][k,j]\ Qc[j,C](X)\ Gl[\mu,a](X)\ T[\sim a,B,C]$

$$L4 := g\, T^{a}{}_{B,C}\, \overline{Qc_{k,B}(X)}\, Qc_{j,C}(X)\, Gl_{\mu,a}(X)\, \left(\gamma^{\mu}\right)_{k,j} \quad (60)$$

> $L5 := g \cdot conjugate(Qt[k,B](X))\ Dgamma[\mu][k,j]\ Qt[j,C](X)\ Gl[\mu,a](X)\ T[\sim a,B,C]$

$$L5 := g\, T^{a}{}_{B,C}\, \overline{Qt_{k,B}(X)}\, Qt_{j,C}(X)\, Gl_{\mu,a}(X)\, \left(\gamma^{\mu}\right)_{k,j} \quad (61)$$

> $L6 := g \cdot conjugate(Qb[k,B](X))\ Dgamma[\mu][k,j]\ Qb[j,C](X)\ Gl[\mu,a](X)\ T[\sim a,B,C]$

$$L6 := g\, T^{a}{}_{B,C}\, \overline{Qb_{k,B}(X)}\, Qb_{j,C}(X)\, Gl_{\mu,a}(X)\, \left(\gamma^{\mu}\right)_{k,j} \quad (62)$$

Then the terms representing the self-interaction of the gluons

> $L7 := g \cdot f[\sim a,\sim b,\sim c]d\_[mu](Gl[nu,a](X)) \cdot Gl[mu,b](X)Gl[nu,c](X)$

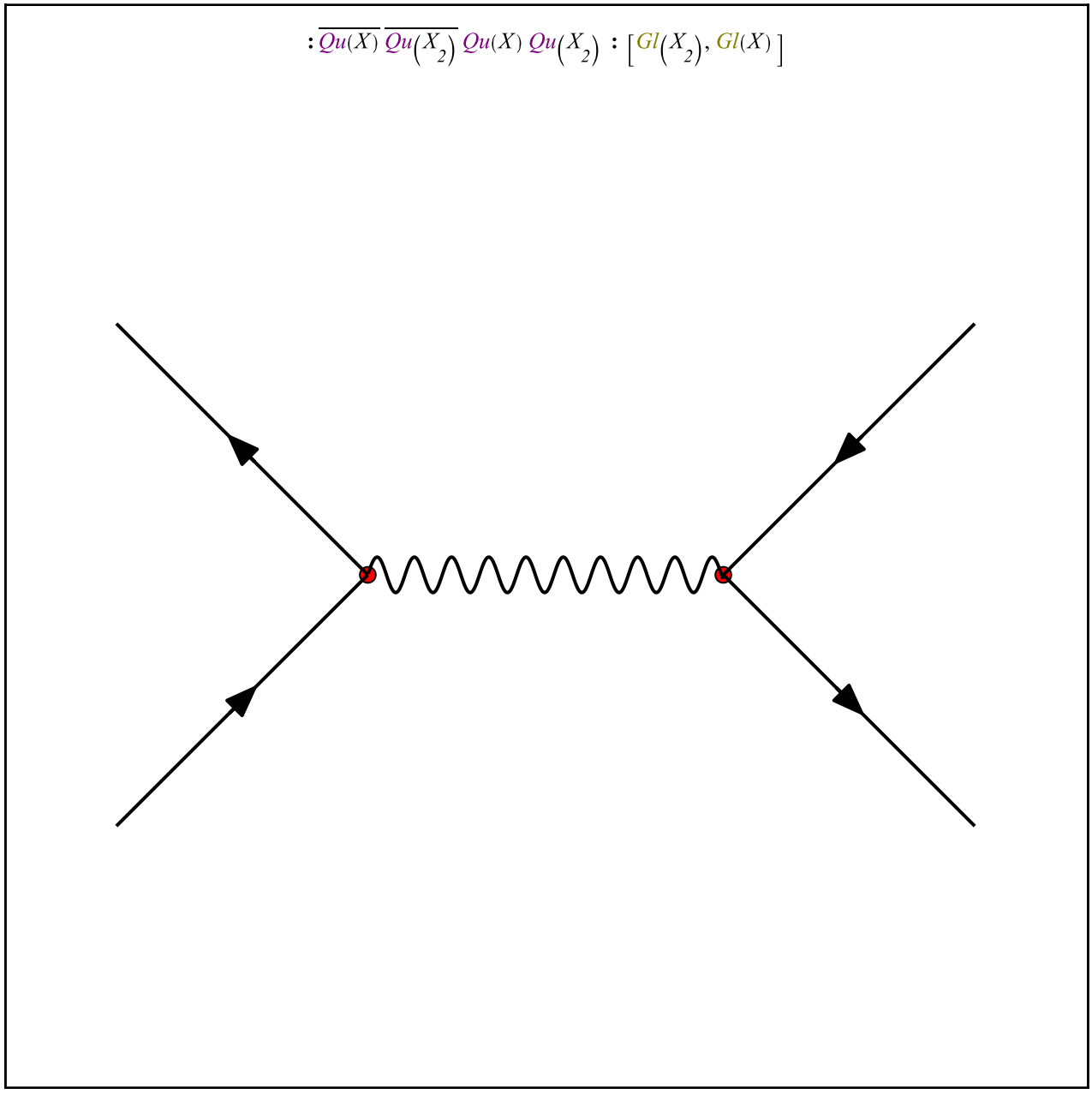
$$L7 := g\, f^{a,b,c}\, \partial_{\mu}\left(Gl_{\nu,a}(X)\right)\, Gl^{\mu}{}_{b}(X)\, Gl^{\nu}{}_{c}(X) \quad (63)$$

> $L8 := g^{2} \cdot f[\sim e,\sim a,\sim b] \cdot f[\sim e,\sim c,\sim d] \cdot Gl[mu,a](X)Gl[lambda,b](X)Gl[mu,c](X)Gl[lambda,d](X)$

$$L8 := g^2 f^{e,a,b} f^{e,c,d} Gl_{\mu,a}(X)\, Gl_{\lambda,b}(X)\, Gl^{\mu}{}_{c}(X)\, Gl^{\lambda}{}_{d}(X) \tag{64}$$

where *L7* involves derivatives of the gluon field. Each of these 8 interaction terms produces a particular vertex, for example, the output in this 2021 version of QCD, equivalent to **(29)** and **(31)** computed with the new StandardModel package's commands, is

> *FeynmanDiagrams*(*L1*, *incomingparticles* = [*Qu*, *conjugate*(*Qu*)], *outgoingparticles* = [*Qu*, *conjugate*(*Qu*)], *numberofloops* = 0, *diagrams*)



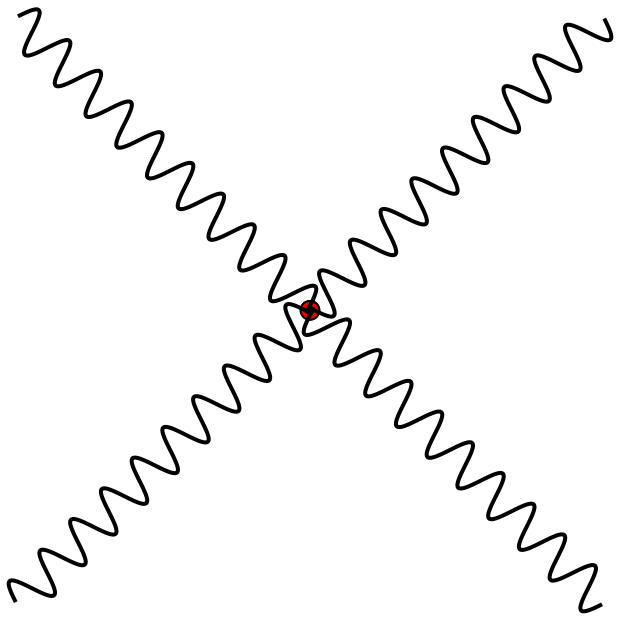
$$-\frac{1}{4\pi^2\left(\left(P_{1_\kappa}+P_{2_\kappa}\right)\left(P_1{}^{\kappa}+P_2{}^{\kappa}\right)+i\epsilon\right)}\left(i\left(\boldsymbol{u}_{Qu}\right)_{l,A}\left(\vec{P}_1\right)\overline{\left(\boldsymbol{v}_{Qu}\right)_{m,E}\left(\vec{P}_2\right)}\right. \tag{65}$$

$$\overline{\left(\boldsymbol{u}_{Qu}\right)_{n,\,F}\left(\vec{P}_3\right)}\left(\boldsymbol{v}_{Qu}\right)_{p,\,G}\left(\vec{P}_4\right) g^2\, T_{c,\,F,\,G}\left(\gamma^{\alpha}\right)_{n,\,p} T_{b,\,E,\,A}\left(\gamma^{\nu}\right)_{m,\,l} g_{\alpha,\,\nu}\,\delta_{b,\,c}\,\delta\Big(-P_3{}^{\beta}-P_4{}^{\beta}+P_1{}^{\beta}+P_2{}^{\beta}\Big)\Big)+\frac{1}{4\,\pi^2\left(\left(P_1{}^{\kappa}-P_3{}^{\kappa}\right)\left(P_{1_{\kappa}}-P_{3_{\kappa}}\right)+\mathrm{i}\,\epsilon\right)}\Big(\mathrm{i}\left(\boldsymbol{u}_{Qu}\right)_{l,\,A}\left(\vec{P}_1\right)$$

$$\overline{\left(\boldsymbol{v}_{Qu}\right)_{m,\,E}\left(\vec{P}_2\right)}\,\overline{\left(\boldsymbol{u}_{Qu}\right)_{n,\,F}\left(\vec{P}_3\right)}\left(\boldsymbol{v}_{Qu}\right)_{p,\,G}\left(\vec{P}_4\right) g^2\, T_{c,\,E,\,G}\left(\gamma^{\alpha}\right)_{m,\,p} T_{b,\,F,\,A}\left(\gamma^{\nu}\right)_{n,\,l} g_{\alpha,\,\nu}\,\delta_{b,\,c}\,\delta\Big(-P_3{}^{\beta}-P_4{}^{\beta}+P_1{}^{\beta}+P_2{}^{\beta}\Big)\Big)$$

> *FeynmanDiagrams*(*L8*, *incomingparticles* = [*Gl*, *Gl*], *outgoingparticles* = [*Gl*, *Gl*], *numberofloops* = 0, *diagrams*)

: $Gl(X)^4$:



$$\frac{1}{\pi^2 \sqrt{E_1 E_2 E_3 E_4}} \left(\frac{i}{8} \left(\boldsymbol{\varepsilon}_{Gl}\right)_{\nu, f}\left(\vec{P}_1\right) \left(\boldsymbol{\varepsilon}_{Gl}\right)_{\alpha, h}\left(\vec{P}_2\right) \overline{\left(\boldsymbol{\varepsilon}_{Gl}\right)_{\beta, a1}\left(\vec{P}_3\right)} \overline{\left(\boldsymbol{\varepsilon}_{Gl}\right)_{\kappa, a2}\left(\vec{P}_4\right)} g^2 \left(\right.\right. \tag{66}$$

$$g^{\beta, \nu} \left(f_{e, a1, a2} f_{e, f, h} + f_{e, f, a2} f_{e, a1, h} + f_{e, a2, a1} f_{e, h, f} + f_{e, a2, f} f_{e, h, a1}\right) g^{\alpha, \kappa} +$$

$$g^{\kappa, \nu} \left(f_{e, h, f} f_{e, a1, a2} + f_{e, h, a2} f_{e, a1, f} + f_{e, a2, a1} f_{e, f, h} + f_{e, f, a1} f_{e, a2, h}\right) g^{\alpha, \beta} + g^{\alpha, \nu}$$

$$g^{\beta, \kappa} \left(f_{e, a2, h} f_{e, a1, f} + f_{e, a2, f} f_{e, a1, h} + f_{e, h, a2} f_{e, f, a1} + f_{e, f, a2} f_{e, h, a1}\right) \left.\left.\right) \delta\left(-P_3{}^{\sigma} - P_4{}^{\sigma} + \right.\right.$$

$$\left.\left. P_1{}^{\sigma} + P_2{}^{\sigma} \right) \right)$$

The total interaction Lagrangian in this 2021 version of QCD, equivalent to the interaction Lagrangian **(27)** constructed with the *StandardModel* commands, is

> $L := L1 + L2 + L3 + L4 + L5 + L6 + L7 + L8$

$$L := g\, T^{a}{}_{B, C}\, \overline{Qu_{k, B}(X)}\, Qu_{j, C}(X)\, Gl_{\mu, a}(X) \left(\gamma^{\mu}\right)_{k, j} + g\, T^{a}{}_{B, C} \tag{67}$$

$$\overline{Qd_{k, B}(X)}\, Qd_{j, C}(X)\, Gl_{\mu, a}(X) \left(\gamma^{\mu}\right)_{k, j} + g\, T^{a}{}_{B, C}\, \overline{Qs_{k, B}(X)}\, Qs_{j, C}(X)\, Gl_{\mu, a}(X) \left(\gamma^{\mu}\right)_{k, j}$$

$$+ g\, T^{a}{}_{B, C}\, \overline{Qc_{k, B}(X)}\, Qc_{j, C}(X)\, Gl_{\mu, a}(X) \left(\gamma^{\mu}\right)_{k, j} + g\, T^{a}{}_{B, C}$$

$$\overline{Qt_{k, B}(X)}\, Qt_{j, C}(X)\, Gl_{\mu, a}(X) \left(\gamma^{\mu}\right)_{k, j} + g\, T^{a}{}_{B, C}\, \overline{Qb_{k, B}(X)}\, Qb_{j, C}(X)\, Gl_{\mu, a}(X) \left(\gamma^{\mu}\right)_{k, j}$$

$$+ g f^{a, b, c} \partial_{\mu}\left(Gl_{\nu, a}(X)\right) Gl^{\mu}{}_{b}(X)\, Gl^{\nu}{}_{c}(X) + g^2 f^{e, a, b} f^{e, c, d}\, Gl_{\mu, a}(X)\, Gl_{\lambda, b}(X)$$

$$Gl^{\mu}{}_{c}(X)\, Gl^{\lambda}{}_{d}(X)$$

Using an Apple laptop with the M1 chip and 32 GB of memory, in Maple 2021.0 released one year ago, the computation at two loops of the process with two incoming bottom quarks (particle and antiparticle) and two outgoing top quarks (particle and antiparticle) takes 45 seconds. In Maple 2022 it takes around 8 seconds

> $st := time(\,) : FeynmanDiagrams(L, incomingparticles = [Qb, conjugate(Qb)],$
> $outgoingparticles = [Qt, conjugate(Qt)], numberofloops = 2) : \mathrm{round}(time(\,) - st) \cdot seconds$

$$8\ seconds \tag{68}$$

If one includes tadpoles and reducible graphs, the computation takes 4 hours in Maple 2021.0, and around 2 minutes in Maple 2022

> $st := time(\,) : FeynmanDiagrams(L, incomingparticles = [Qb, conjugate(Qb)],$
> $outgoingparticles = [Qt, conjugate(Qt)], numberofloops = 2, includetadpoles = true,$
> $reduciblegraphs = true) : \mathrm{round}\left(\frac{(time(\,) - st)}{60.}\right) \cdot minutes$

$$2\ minutes \tag{69}$$

This same process at three loops, even without tadpoles and reducible graphs is out of reach in Maple 2021. In Maple 2022, that computation at three loops completes in around 20 minutes

> $st := time(\,) : FeynmanDiagrams(L, incomingparticles = [Qb, conjugate(Qb)],$
$outgoingparticles = [Qt, conjugate(Qt)], numberofloops = 3) : \mathrm{round}\left(\frac{(time(\,) - st)}{60.}\right)$
$\cdot minutes$

$$20\ minutes \tag{70}$$

Drawing Feynman diagrams with 4 and 5 vertices

Load the package, set three coordinate systems and set ϕ to represent a quantum operator

> $restart; with(Physics) :$

> $Setup(mathematicalnotation = true, coordinates = [X, Y, Z], quantumoperators = \phi)$

Systems of spacetime coordinates are: $\{X = (x1, x2, x3, x4), Y = (y1, y2, y3, y4), Z = (z1, z2, z3, z4)\}$

$$[coordinatesystems = \{X, Y, Z\}, mathematicalnotation = true, quantumoperators = \{\phi\}] \tag{71}$$

Let L be the interaction Lagrangian

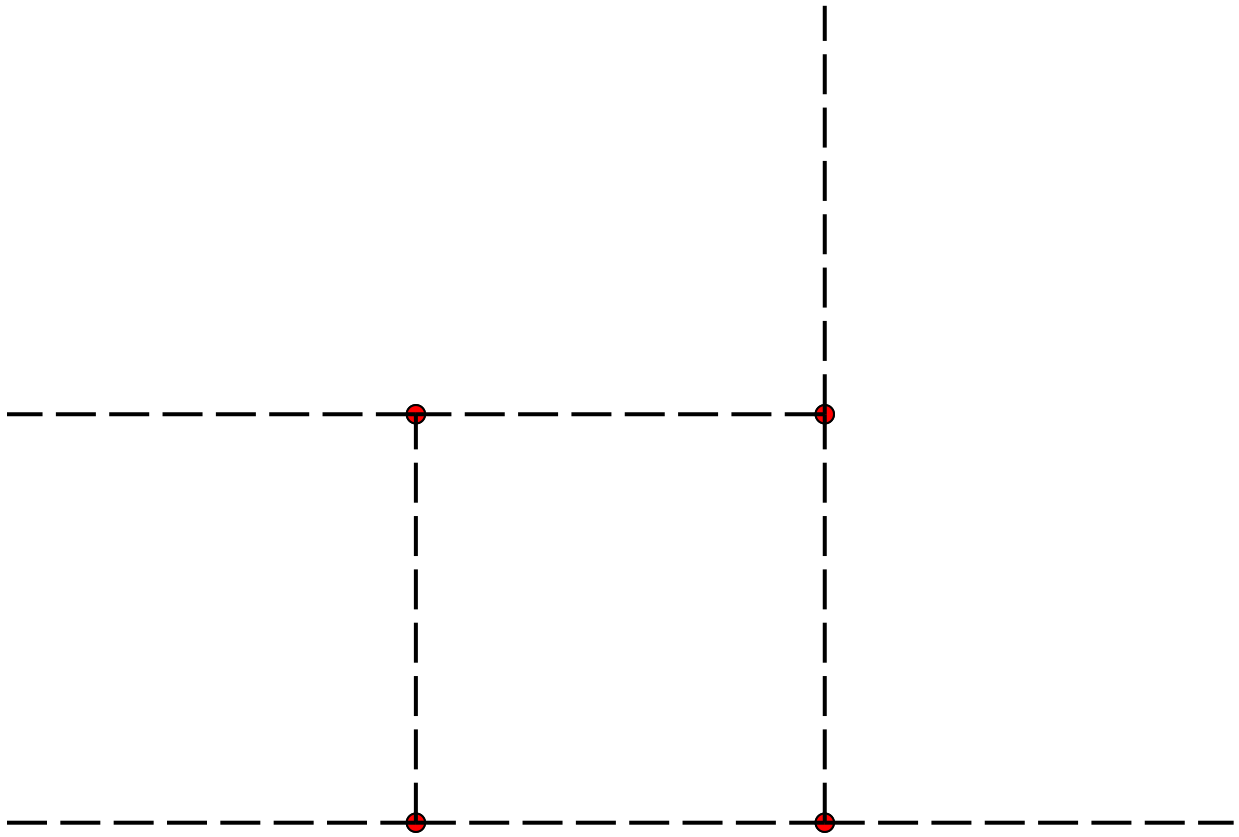
> $L := \lambda \phi(X)^3$

$$L := \lambda \phi(X)^3 \tag{72}$$

New in Maple 2022, diagrams with 4 and 5 vertices are now drawn. The following is the term of the scattering matrix in coordinates representation with up to four external legs - i.e. related to processes involving four particles in total, adding the *incoming* and *outgoing* - and only the fourth term ($numberofvertices = 4$)

> $FeynmanDiagrams(L, numberofexternallegs = 4, diagrams, numberofvertices = 4)$

$$:\phi(X_4)\,\phi(X)\,\phi(Y)\,\phi(Z):\left[\phi(X_4),\phi(X)\right]\left[\phi(X_4),\phi(Z)\right]\left[\phi(X),\phi(Y)\right]\left[\phi(Y),\phi(Z)\right]$$



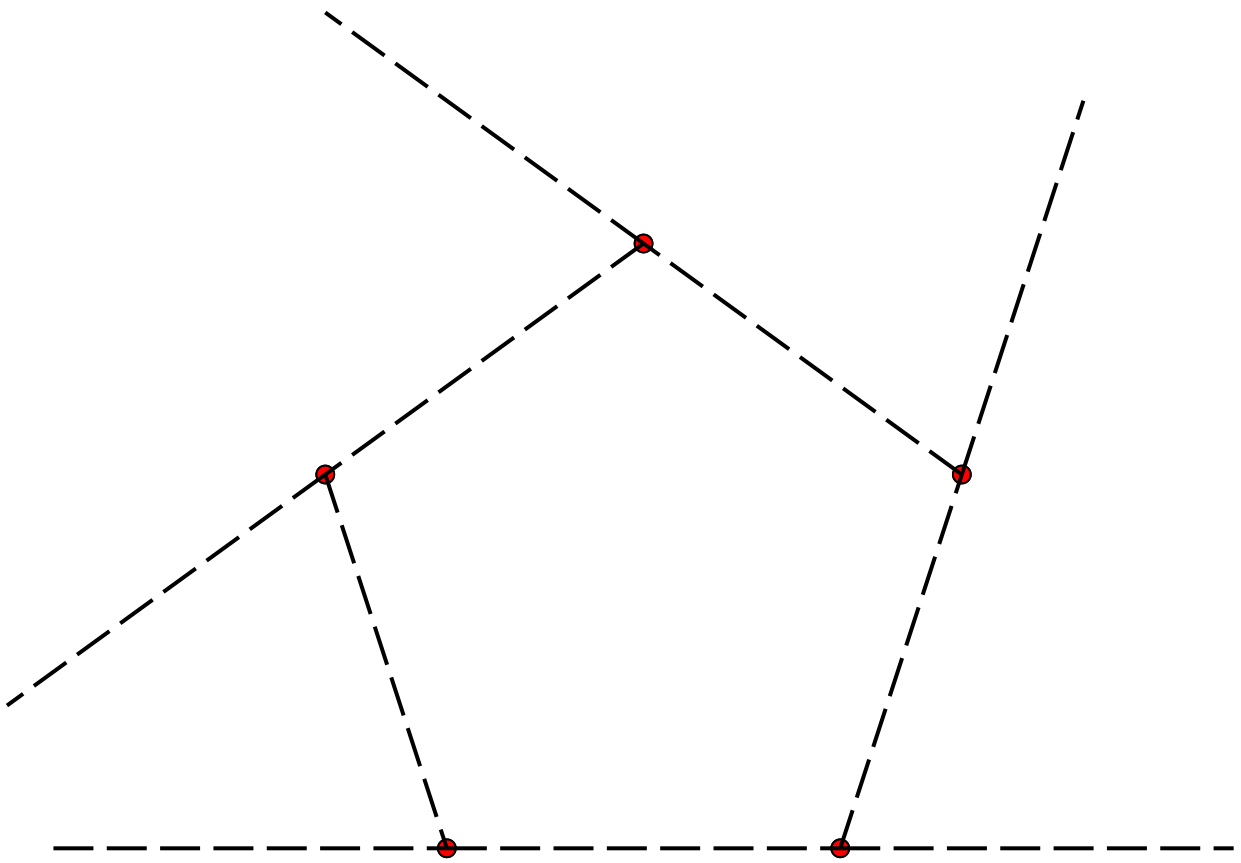
Symmetry factor = 3888

$$\frac{\Gamma^4}{4!}\int\int\int\int\int 3888\,\lambda^4\,:\phi(X_4)\,\phi(X)\,\phi(Y)\,\phi(Z):\left[\phi(X_4),\phi(X)\right]\left[\phi(X_4),\phi(Z)\right]\left[\phi(X),\phi(Y)\right]\left[\phi(Y),\phi(Z)\right]dX^4dY^4dZ^4dX_4{}^4 \quad (73)$$

The same computation, again in coordinates representation with five external legs - processes involving five particles in total, adding the *incoming* and *outgoing*

> *FeynmanDiagrams*(*L*, *numberofvertices* = 5, *numberofexternallegs* = 5, *diagrams*)

$$:\phi(X_4)\,\phi(X_5)\,\phi(X)\,\phi(Y)\,\phi(Z):\left[\phi(X_4),\phi(X)\right]\left[\phi(X_4),\phi(Y)\right]\left[\phi(X_5),\phi(X)\right]\left[\phi(X_5),\phi(Z)\right]\left[\phi(Y),\phi(Z)\right]$$



Symmetry factor = 93312

$$\frac{\Gamma^5}{5!}\int\int\int\int\int 93312\,\lambda^5\,:\phi(X_4)\,\phi(X_5)\,\phi(X)\,\phi(Y)\,\phi(Z):\left[\phi(X_4),\phi(X)\right]\left[\phi(X_4),\phi(Y)\right]\left[\phi(X_5),\phi(X)\right]\left[\phi(X_5),\phi(Z)\right]\left[\phi(Y),\phi(Z)\right]dX^4dY^4dZ^4dX_4{}^4dX_5{}^4 \quad (74)$$

For a particular scattering process of this type, consider for instance

incoming = [ϕ, ϕ, ϕ], *outgoing* = [ϕ, ϕ], the corresponding input would be *FeynmanDiagrams*(*L*, *incoming* = [phi, phi, phi], *outgoing* = [phi, phi], *numberofvertices* = 5, *numberofloops* = 1, *diagrams*).

A more involved example with the $\lambda\,\phi^4$ model, cases of 4 and 5 vertices. To understand the drawings, in the case of 4 vertices there are 4 red dots in the corners of a square, then there are 6 internal lines: the perimeter and the two diagonals.

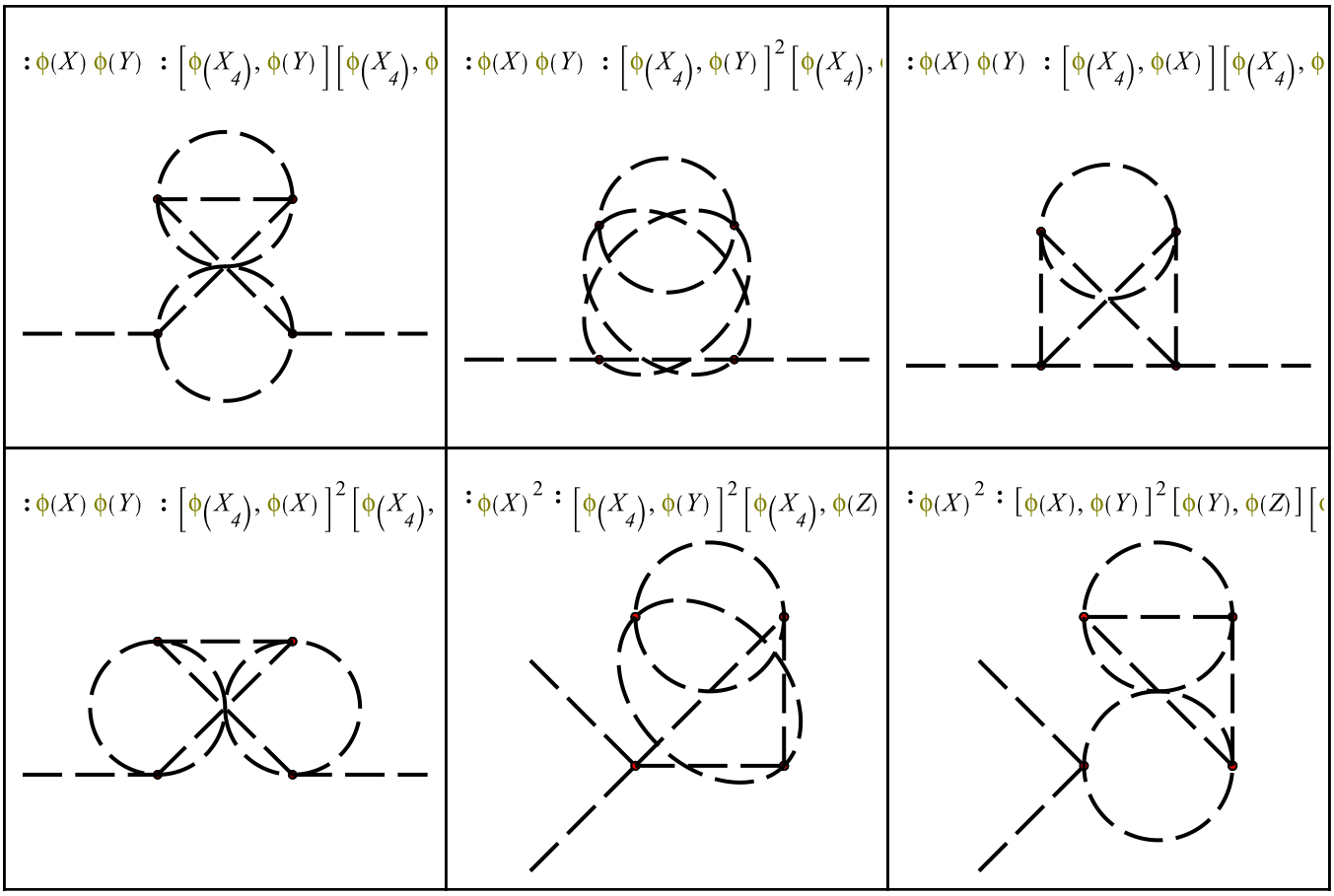
> $L := \lambda\,\phi(X)^4$

$$L := \lambda\,\phi(X)^4 \tag{75}$$

The scattering amplitude in momentum representation for one incoming and one outgoing particles and four loops; to see the algebraic mathematical expression of the amplitude, remove the `:` at the end of the input line

> $FeynmanDiagrams(L, incoming = [\phi], outgoing = [\phi], loops = 4, numberofvertices = 4, diagrams)$:

** Partial match of 'loops' against keyword 'numberofloops'*



For a more involved example at five loops consider the input line
$FeynmanDiagrams(L, incoming = [\phi], outgoing = [\phi], loops = 5, numberofvertices = 5, diagrams)$

>

FeynmanIntegral module with 9 new commands

FeynmanIntegral, introduced in Maple 2021, is both a command and a package of commands for the

computation of Feynman integrals, i.e. the (loop) integrals that appear in quantum field theory when performing perturbative calculations with the S-matrix in momentum representation. In Maple 2021.0, *FeynmanIntegral* entered the Maple library with two commands: *Evaluate* to evaluate the integrals, and *Parametrize* to parametrize them.

In Maple 2022, the command and the package got significantly extended, both regarding the representation and computation capabilities. As a command, FeynmanIntegral can now compute a Feynman integral using dimensional regularization, rewriting the integrand using tensor reduction, Feynman parameters, and expanding in the dimensional parameter ε. As a package, Maple 2022 extended the FeynmanIntegral (overview) package with the addition of 9 new commands for performing most of the steps of the computation of these integrals.

epsilon	Evaluate	ExpandDimension	FeynmanIntegral
FromAbstractRepresentation	Parametrize	Series	TensorBasis
TensorReduce	ToAbstractRepresentation	varepsilon	

Brief description of the commands of the FeynmanIntegral package in Maple 2022

- epsilon is the same as *FeynmanDiagrams:-epsilon* and is used to express the prescription used to integrate in the complex p^0 plane.
- Evaluate evaluates the Feynman integrals of a given expression, typically the output of the FeynmanDiagrams command, by parametrizing each of those integrals, optionally returning the intermediate steps of the computation or expanding the dimension around $d = 4 - 2\,\varepsilon$ keeping terms up to $O(\varepsilon)$.
- ExpandDimension expands the d-dimensional result returned by **Evaluate** around $d = 4 - 2\,\varepsilon$ keeping terms up to $O(\varepsilon)$.
- FromAbstractRepresentation returns the standard form of Feynman integrals passed in abstract form. This command is the reverse of ToAbstractRepresentation.
- Parametrize replaces the propagators within a Feynman integral by integrals on Feynman or alpha parameters.
- Series expands in series, like the series command, but strictly returning results up to order $O(n)$ regardless of the existence of terms with negative powers.
- TensorBasis given a list of external momenta and a list of spacetime indices, constructs a complete basis of tensorial structures using the external momenta and the metric $g_{\mu,\nu}$.

- TensorReduce expresses integrals with loop momenta in the numerator in terms of scalar integrals (with no loop momenta in the numerator).
- ToAbstractRepresentation represents Feynman integrals in an abstract form suitable for performing the tensor reduction of tensor Feynman integrals. The abstract form is suitable for performing the tensor reduction of Feynman integrals implemented in TensorReduce.
- varepsilon represents the dimensional parameter used in the dimension regularization approach.

Examples

```
> restart; with(Physics) :
```

```
> with(FeynmanIntegral)
```

$$[Evaluate, ExpandDimension, FromAbstractRepresentation, Parametrize, Series, SumLookup, TensorBasis, TensorReduce, ToAbstractRepresentation, \epsilon, \varepsilon] \quad (76)$$

To remain closer to textbook notation, display the imaginary unit with a lowercase i

```
> interface(imaginaryunit = i) :
```

The simplest case of a massive ϕ field, the integral containing two propagators and one external momentum P_1^{μ} to which corresponds the mass m_1.

```
> %FeynmanIntegral( p_I[~mu] / ((p_I^2 - m_phi^2 + i*epsilon)*((p_I - P_I)^2 - m_I^2 + i*epsilon)), p_I )
```

$$\int \frac{p_I{}^{\mu}}{\left(p_I{}^2 - m_{\phi}{}^2 + i\,\epsilon\right)\left(\left(p_I - P_I\right)^2 - m_I{}^2 + i\,\epsilon\right)}\, dp_I{}^4 \quad (77)$$

The reduction of this *tensor* integral to a linear combination of *scalar* Feynman integrals all in one go:

```
> (77) = TensorReduce( (77) )
```

$$\int \frac{p_I{}^{\mu}}{\left(p_I{}^2 - m_{\phi}{}^2 + i\,\epsilon\right)\left(\left(p_I - P_I\right)^2 - m_I{}^2 + i\,\epsilon\right)}\, dp_I{}^4 = -\frac{1}{2\left(P_I \cdot P_I\right)}\left(P_I{}^{\mu}\left(\left(m_I{}^2 - m_{\phi}{}^2 - P_I \cdot P_I\right)\int \frac{1}{\left(p_I{}^2 - m_{\phi}{}^2 + i\,\epsilon\right)\left(\left(p_I - P_I\right)^2 - m_I{}^2 + i\,\epsilon\right)}\, dp_I{}^4 + \int \frac{1}{p_I{}^2 - m_{\phi}{}^2 + i\,\epsilon}\, dp_I{}^4 - \int \frac{1}{\left(p_I - P_I\right)^2 - m_I{}^2 + i\,\epsilon}\, dp_I{}^4\right)\right) \quad (78)$$

By design, the reduction process *does not evaluate the integrals* so that one can follow the process clearly. The evaluation can be performed next by passing this result to Evaluate

> **(77)** = *Evaluate*(*rhs*(**(78)**))

$$\int \frac{p_1{}^{\mu}}{\left(p_1{}^2 - m_\phi{}^2 + \mathrm{i}\,\epsilon\right)\left(\left(p_1 - P_1\right)^2 - m_1{}^2 + \mathrm{i}\,\epsilon\right)}\, \mathrm{d}p_1{}^4 = -\frac{1}{2\left(P_1 \cdot P_1\right)}\left(P_1{}^{\mu}\left(-\mathrm{i}\left(m_1{}^2 - m_\phi{}^2 - P_1 \cdot P_1\right)\pi^{2-\varepsilon}\left(\sum_{n=0}^{\infty}\sum_{n_1=0}^{\infty}\left(-\frac{\left(-m_1{}^2 + m_\phi{}^2\right)^n \Gamma\left(n + n_1 + 1\right) P_1{}^{2 n_1}\, \Gamma\left(\varepsilon + n + n_1\right) m_\phi{}^{-2 n_1 - 2\varepsilon - 2 n}}{\Gamma\left(2 n_1 + n + 2\right)\Gamma(1 + n)}\right)\right)\right.\right.\right.$$

$$\left.\left.\left. - \mathrm{i}\,\pi^{2-\varepsilon} m_\phi{}^{2-2\varepsilon}\Gamma(-1 + \varepsilon) + \mathrm{i}\,\pi^{2-\varepsilon} m_1{}^{2-2\varepsilon}\Gamma(-1 + \varepsilon)\right)\right)\right) \qquad \textbf{(79)}$$

Note also that Evaluate automatically calls TensorReduce to perform the reduction of tensor integrals when that is the case. So, passing the Feynman integral directly to Evaluate, skipping the interactive TensorReduce step, results in the same

> *Evaluate*(**(77)**)

$$-\frac{1}{2\left(P_1 \cdot P_1\right)}\left(P_1{}^{\mu}\left(-\mathrm{i}\left(m_1{}^2 - m_\phi{}^2 - P_1 \cdot P_1\right)\pi^{2-\varepsilon}\left(\sum_{n=0}^{\infty}\sum_{n_1=0}^{\infty}\left(-\frac{\left(-m_1{}^2 + m_\phi{}^2\right)^n \Gamma\left(n + n_1 + 1\right) P_1{}^{2 n_1}\, \Gamma\left(\varepsilon + n + n_1\right) m_\phi{}^{-2 n_1 - 2\varepsilon - 2 n}}{\Gamma\left(2 n_1 + n + 2\right)\Gamma(1 + n)}\right)\right)\right.\right.\right.$$

$$\left.\left.\left. - \mathrm{i}\,\pi^{2-\varepsilon} m_\phi{}^{2-2\varepsilon}\Gamma(-1 + \varepsilon) + \mathrm{i}\,\pi^{2-\varepsilon} m_1{}^{2-2\varepsilon}\Gamma(-1 + \varepsilon)\right)\right)\right) \qquad \textbf{(80)}$$

Back to the reduction process, this is how the integral is processed one step at a time. First, check the abstract representation that will be used in the output step by step:

> *ToAbstractRepresentation*(**(77)**)

$$\mathbb{T}^{\mu}\left(2, 0, m_\phi{}^2, -P_1, m_1{}^2, p_1, 0\right) \qquad \textbf{(81)}$$

In this output we see the integral has 2 propagators, the first one has 0 external momentum (i.e. none) and mass $m_\phi{}^2$. The second propagator has external momentum $-P_1$ to which corresponds the mass m_1. Finally the loop momentum integration variable is p_1 and the last operand, in this example equal to 0

means there are no contracted powers of p_1, the loop integration variable, in the numerator of the integrand. To retrieve the non-abstract form from the abstract one you can use

> *FromAbstractRepresentation*(**(81)**)

$$\int \frac{p_1{}^{\mu}}{\left(p_1{}^2 - m_\phi{}^2 + \mathrm{i}\,\epsilon\right)\left(\left(p_1 - P_1\right)^2 - m_1{}^2 + \mathrm{i}\,\epsilon\right)}\, \mathrm{d}p_1{}^4 \tag{82}$$

The first step of the Passarino-Veltman reduction, the main equation

> *TensorReduce*(**(77)**, *step* = 1)

** Partial match of 'step' against keyword 'outputstep'*

$$\int \frac{p_1{}^{\mu}}{\left(p_1{}^2 - m_\phi{}^2 + \mathrm{i}\,\epsilon\right)\left(\left(p_1 - P_1\right)^2 - m_1{}^2 + \mathrm{i}\,\epsilon\right)}\, \mathrm{d}p_1{}^4 = C_1\, P_1{}^{\mu} \tag{83}$$

The right-hand side contains only one element. That is so because the tensor basis for this problem, where there is only one free spacetime index in the numerator of the integral and only one external momentum P_1 is given by just

> *TensorBasis*([P_1], [~*mu*])

$$\left[P_1{}^{\mu}\right] \tag{84}$$

The second step

> *TensorReduce*(**(77)**, *step* = 2)

** Partial match of 'step' against keyword 'outputstep'*

$$\left[\, P_{1\mu}\, \mathbb{T}^{\mu}\left(2, 0, m_\phi{}^2, -P_1, m_1{}^2, p_1, 0\right) = P_{1\mu}\, C_1\, P_1{}^{\mu} \,\right] \tag{85}$$

The third step only represents the scalar products in the right-hand side in a more convenient form

> *TensorReduce*(**(77)**, *step* = 3)

** Partial match of 'step' against keyword 'outputstep'*

$$\left[\, P_{1\mu}\, \mathbb{T}^{\mu}\left(2, 0, m_\phi{}^2, -P_1, m_1{}^2, p_1, 0\right) = C_1\, \left(P_1 \cdot P_1\right) \,\right] \tag{86}$$

The fourth step is the most important one, where the actual reduction to scalar integrals, represented in abstract form, is performed

> *TensorReduce*(**(77)**, *step* = 4)

** Partial match of 'step' against keyword 'outputstep'*

(87)

$$\left[-\frac{\mathbb{T}[\,]\left(1, 0, m_\phi^{\,2}, p_1, 0\right)}{2} + \frac{\mathbb{T}[\,]\left(1, -P_1, m_1^{\,2}, p_1, 0\right)}{2} + \frac{\left(P_1 \cdot P_1 - m_1^{\,2} + m_\phi^{\,2}\right) \mathbb{T}[\,]\left(2, 0, m_\phi^{\,2}, -P_1, m_1^{\,2}, p_1, 0\right)}{2} = C_1 \left(P_1 \cdot P_1\right) \right] \tag{87}$$

To see this result in standard representation you can use

> *FromAbstractRepresentation*(**(87)**)

$$\left[-\frac{\int \frac{1}{p_1^{\,2} - m_\phi^{\,2} + \mathrm{i}\,\epsilon}\, \mathrm{d}p_1^{\,4}}{2} + \frac{\int \frac{1}{\left(p_1 - P_1\right)^2 - m_1^{\,2} + \mathrm{i}\,\epsilon}\, \mathrm{d}p_1^{\,4}}{2} + \frac{\left(P_1 \cdot P_1 - m_1^{\,2} + m_\phi^{\,2}\right) \int \frac{1}{\left(p_1^{\,2} - m_\phi^{\,2} + \mathrm{i}\,\epsilon\right)\left(\left(p_1 - P_1\right)^2 - m_1^{\,2} + \mathrm{i}\,\epsilon\right)}\, \mathrm{d}p_1^{\,4}}{2} = C_1 \left(P_1 \cdot P_1\right) \right] \tag{88}$$

The fifth step processes this output by solving for the C_n coefficients, expressing them in terms of the scalar integrals of step 4.

> *TensorReduce*(**(77)**, *step* = 5)

** Partial match of 'step' against keyword 'outputstep'*

$$\int \frac{p_1^{\,\mu}}{\left(p_1^{\,2} - m_\phi^{\,2} + \mathrm{i}\,\epsilon\right)\left(\left(p_1 - P_1\right)^2 - m_1^{\,2} + \mathrm{i}\,\epsilon\right)}\, \mathrm{d}p_1^{\,4} = C_1\, P_1^{\,\mu} \textbf{ where } \left\{ C_1 = \frac{1}{2\left(P_1 \cdot P_1\right)} \left(- \mathbb{T}[\,]\left(2, 0, m_\phi^{\,2}, -P_1, m_1^{\,2}, p_1, 0\right) m_1^{\,2} + \mathbb{T}[\,]\left(2, 0, m_\phi^{\,2}, -P_1, m_1^{\,2}, p_1, 0\right) m_\phi^{\,2} + \left(P_1 \cdot P_1\right) \mathbb{T}[\,]\left(2, 0, m_\phi^{\,2}, -P_1, m_1^{\,2}, p_1, 0\right) - \mathbb{T}[\,]\left(1, 0, m_\phi^{\,2}, p_1, 0\right) + \mathbb{T}[\,]\left(1, -P_1, m_1^{\,2}, p_1, 0\right) \right) \right\} \tag{89}$$

The sixth step combines this result inserting, in the output of step 1, the values of the C_n

> *TensorReduce*(**(77)**, *step* = 6)

** Partial match of 'step' against keyword 'outputstep'*

$$\int \frac{p_1^{\,\mu}}{\left(p_1^{\,2} - m_\phi^{\,2} + \mathrm{i}\,\epsilon\right)\left(\left(p_1 - P_1\right)^2 - m_1^{\,2} + \mathrm{i}\,\epsilon\right)}\, \mathrm{d}p_1^{\,4} = \frac{1}{2\left(P_1 \cdot P_1\right)} \left(\left(- \mathbb{T}[\,]\left(2, 0, m_\phi^{\,2}, -P_1, m_1^{\,2}, p_1, 0\right) m_1^{\,2} + \mathbb{T}[\,]\left(2, 0, m_\phi^{\,2}, -P_1, m_1^{\,2}, p_1, 0\right) m_\phi^{\,2} + \left(P_1 \cdot P_1\right) \mathbb{T}[\,]\left(2, 0, m_\phi^{\,2}, -P_1, m_1^{\,2}, \right. \right. \tag{90}$$

$$p_1, 0\Big) - \mathbb{T}[\,]\Big(1, 0, m_\phi^2, p_1, 0\Big) + \mathbb{T}[\,]\Big(1, -P_1, m_1^2, p_1, 0\Big)\Big)\, P_1{}^{\mu}\Big)$$

Finally, either passing **step = 7** or omitting the **step = ...** altogether, the whole reduction is performed as done at the beginning (see equation **(3)**).

An example with two free indices

> *%FeynmanIntegral*$\left(\dfrac{p_1[\sim mu]\, p_1[\sim nu]}{\left(p_1^2 - m_1^2 + \mathrm{I}\,\epsilon\right)\left(\left(p_1 - P_1\right)^2 - m_1^2 + i\,\epsilon\right)}, p_1\right)$

$$\int \frac{p_1{}^{\mu}\, p_1{}^{\nu}}{\left(p_1^2 - m_1^2 + \mathrm{i}\,\epsilon\right)\left(\left(p_1 - P_1\right)^2 - m_1^2 + \mathrm{i}\,\epsilon\right)}\, \mathrm{d}p_1{}^4 \qquad \textbf{(91)}$$

The reduction of this tensor integral to scalar integrals, all in one go, is given by

> **(91)** = *TensorReduce*(**(91)**)

$$\int \frac{p_1{}^{\mu}\, p_1{}^{\nu}}{\left(p_1^2 - m_1^2 + \mathrm{i}\,\epsilon\right)\left(\left(p_1 - P_1\right)^2 - m_1^2 + \mathrm{i}\,\epsilon\right)}\, \mathrm{d}p_1{}^4 = \frac{1}{(-12 + 8\,\varepsilon)\,(P_1 \cdot P_1)^2}\Bigg(4\,(P_1 \cdot P_1)\left(\frac{g^{\mu,\nu}\,(P_1 \cdot P_1)^2}{4} + \left(\frac{P_1{}^{\nu}\,(-2+\varepsilon)\,P_1{}^{\mu}}{2} - m_1^2\, g^{\mu,\nu}\right)(P_1 \cdot P_1) + m_1^2\, P_1{}^{\mu}\, P_1{}^{\nu}\right)\int \frac{1}{\left(p_1^2 - m_1^2 + \mathrm{i}\,\epsilon\right)\left(\left(p_1 - P_1\right)^2 - m_1^2 + \mathrm{i}\,\epsilon\right)}\, \mathrm{d}p_1{}^4 + \Big(-2\, g^{\mu,\nu}\,(P_1 \cdot P_1)^2 + \big(4\, P_1{}^{\nu}\,(-1+\varepsilon)\,P_1{}^{\mu} - m_1^2\, g^{\mu,\nu}\big)(P_1 \cdot P_1) - 2\, m_1^2\, P_1{}^{\mu}\, P_1{}^{\nu}\,(-2+\varepsilon)\Big)\int \frac{1}{\left(p_1 - P_1\right)^2 - m_1^2 + \mathrm{i}\,\epsilon}\, \mathrm{d}p_1{}^4 + 2\left(\frac{g^{\mu,\nu}\,(P_1 \cdot P_1)}{2} + P_1{}^{\nu}\,(-2+\varepsilon)\,P_1{}^{\mu}\right)\int \frac{1}{p_1^2 - m_1^2 + \mathrm{i}\,\epsilon}\, \mathrm{d}p_1{}^4\left(P_1^2 + m_1^2 - P_1 \cdot P_1\right)\Bigg) \qquad \textbf{(92)}$$

Integral Vector Calculus and Parametrization of curves, surfaces and volumes

Four new commands were added to the Physics:-Vectors package, implementing the parametrization of curves, surfaces and volumes, as well as the computation of path, surface and volume vector integrals. Those are integrals where the integrand is a scalar or vector function, and the computation is done from any description (algebraic, parametric, vectorial) of the region of integration - a path, surface or volume.

There are three kinds of line or path integrals:

$$\left(\int_{\vec{A}}^{\vec{B}} F \, \mathrm{d}\vec{r}\right)_{path = C}$$

$$\left(\int_{\vec{A}}^{\vec{B}} \vec{F} \cdot \mathrm{d}\vec{r}\right)_{path = C}$$

$$\left(\int_{\vec{A}}^{\vec{B}} \vec{F} \times \mathrm{d}\vec{r}\right)_{path = C}$$

where $\vec{A}$ and $\vec{B}$ are points in space, the limits of integration, that belong to the curve C over which the integral is performed. In the first integral, F is a scalar function and the result of the integration is thus a vector. In the second and third integrals the integrand $\vec{F}$ is a vector function, so that the dot product $\vec{F} \cdot d\vec{r}$ is a scalar and so is the integral, while in the third one $\vec{F} \times d\vec{r}$ is a vector and so is its integration over the region C.

Likewise, there are three kinds of surface integrals

$$\left(\int F \, \mathrm{d}\vec{S}\right)_{surface = C,\ parameters = [u = a..b,\ v = c..d]}$$

$$\left(\int \vec{F} \cdot \mathrm{d}\vec{S}\right)_{surface = C,\ parameters = [u = a..b,\ v = c..d]}$$

$$\left(\int \vec{F} \times \mathrm{d}\vec{S}\right)_{surface = C,\ parameters = [u = a..b,\ v = c..d]}$$

and two types of volume integrals

$$\left(\int F \, \mathrm{d}V\right)_{volume = C,\ parameters = [u = a..b,\ v = c..d,\ w = f..g]}$$

$$\left(\int \vec{F}\, \mathrm{d}V\right)_{volume = C,\ parameters = [u = a..b,\ v = c..d,\ w = f..g]}$$

The line element $d\vec{r}$ in path integrals is expressed in terms of a parameter t as

$$d\vec{r} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{r}(t)\right) dt$$

Using indexed notation for derivatives $\vec{r}_u = \frac{\partial}{\partial u}\vec{r}(u, v)$, the surface element $d\vec{S}$ in surface integrals is expressed in terms of parameters as

$$d\vec{S} = \vec{r}_u \times \vec{r}_v\, du\, dv$$

and the volume element dV in volume integrals as

$$dV = \vec{r}_u \cdot \left(\vec{r}_v \times \vec{r}_w\right) du\, dv\, dw$$

The integrals in the three cases are computed by first expressing the integrand and the integration element in terms of the parameters using the parametric equations derived with ParametrizeCurve, ParametrizeSurface and ParametrizeVolume, then performing the vector product operations, then the integration.

Vectorial Path integrals

```
> restart; with(Physics:-Vectors);
```

[&x, `+`, `.`, ChangeBasis, ChangeCoordinates, Component, Curl, DirectionalDiff, Divergence, Gradient, Identify, Laplacian, ∇, Norm, ParametrizeCurve, ParametrizeSurface, ParametrizeVolume, Setup, diff, int] **(93)**

Consider the following scalar function F, path C and integration limits $\vec{A}$ and $\vec{B}$

```
> F := x*y
```

$$F := x\,y \quad \textbf{(94)}$$

```
> C := {y = x^2, z = 0}
```

$$C := \{y = x^2, z = 0\} \quad \textbf{(95)}$$

```
> A_ := [0, 0, 0]
```

$$\vec{A} := [0, 0, 0] \quad \textbf{(96)}$$

```
> B_ := [1, 1, 0]
```

$$\vec{B} := [1, 1, 0] \quad \textbf{(97)}$$

The line or path integral, shown here in inert form on the left-hand side and active, computed to the end on the right-hand side, is

> $(Int = int)(F, r\_ = A\_..B\_, path = C)$

$$\left(\int_{[0,\,0,\,0]}^{[1,\,1,\,0]} x\,y\,d\vec{r}\right)_{path = \{y = x^2,\ z = 0\}} = \frac{2\,\hat{j}}{5} + \frac{\hat{i}}{4} \qquad \textbf{(98)}$$

The output on the right-hand side is a vector. Within int, to perform this integration the curve $C = \{y = x^2, z = 0\}$ is first parametrized using ParametrizeCurve

> $ParametrizeCurve(C)$

$$[x(t) = t, y(t) = t^2, z(t) = 0], t \qquad \textbf{(99)}$$

The output above is a sequence, first the parametric equations as an ordered list (order $[x, y, z]$) then the parameter, in this case t. For the formulation of the integral to make sense, the limits of integration $\vec{A}$ and $\vec{B}$ must belong to this curve, i.e. satisfy the parametric equations for some value of the parameter, in this case $t = 0$ and $t = 1$

> $A\_ = Eval(\textbf{(99)}[1], t = 0)$

$$[0, 0, 0] = [x(t) = t, y(t) = t^2, z(t) = 0]\Big|_{t = 0} \qquad \textbf{(100)}$$

> $B\_ = Eval(\textbf{(99)}[1], t = 1)$

$$[1, 1, 0] = [x(t) = t, y(t) = t^2, z(t) = 0]\Big|_{t = 1} \qquad \textbf{(101)}$$

To see the integral after being parametrized and before performing the integration you can use the option *inert*

> $(Int = int)(F, r\_ = A\_..B\_, path = C, inert)$

$$\left(\int_{[0,\,0,\,0]}^{[1,\,1,\,0]} x\,y\,d\vec{r}\right)_{path = \{y = x^2,\ z = 0\},\ inert} = \int_0^1 \left(2\,\hat{j}\,t^4 + \hat{i}\,t^3\right) dt \qquad \textbf{(102)}$$

Since there is a relation one-to-one between the integration limits $\vec{A}$ and $\vec{B}$ and the parameter's range, instead of indicating $\vec{A}$ and $\vec{B}$ you can also indicate the range of t itself, getting the same result **(98)**

> $(Int = int)(F, r\_, path = C, parameter = \{t = 0..1\})$

$$\left(\int x\,y\,d\vec{r}\right)_{path = \{y = x^2,\ z = 0\},\ parameter = \{t = 0..1\}} = \frac{2\,\hat{j}}{5} + \frac{\hat{i}}{4} \qquad \textbf{(103)}$$

When the parameter's range is passed, you can also use a shortcut notation, passing the second argument as an equation **r_ = C**, making more explicit that **r_** is the vector representation of the region of integration **C**, so the line element $d\vec{r}$ is the differential of the vectorial parametric representation of **C**. In that case, indicating **path = C** is redundant and can be omitted.

> $(Int = int)(F, r\_ = C, parameter = \{t = 0..1\})$

$$\left(\int x\,y\,d\vec{r}\right)_{path = \{y = x^2,\, z = 0\},\, parameter = \{t = 0..1\}} = \frac{2\,\hat{j}}{5} + \frac{\hat{i}}{4} \quad \textbf{(104)}$$

The integration path and the limits of integration can be expressed in vector notation as well

> $C\_ := x\_i + x^2\_j$

$$\vec{C} := x^2\,\hat{j} + x\,\hat{i} \quad \textbf{(105)}$$

> $A\_ := 0$

$$\vec{A} := 0 \quad \textbf{(106)}$$

> $B\_ := \_i + \_j$

$$\vec{B} := \hat{i} + \hat{j} \quad \textbf{(107)}$$

> $(Int = int)(F, r\_ = A\_..B\_, path = C\_)$

$$\left(\int_0^{\hat{i}+\hat{j}} x\,y\,d\vec{r}\right)_{path = x^2\,\hat{j} + x\,\hat{i}} = \frac{2\,\hat{j}}{5} + \frac{\hat{i}}{4} \quad \textbf{(108)}$$

Vectorial Surface and Volume integrals

The case of surface and volume integrals is analogous to that of line (path) integrals, but for two things: instead of one, there are two or three parameters, and instead of indicating integration limits, it is required that you indicate the parameter's ranges.

The following C represents the surface of a sphere of radius a centered at the origin

> $with(Physics, Assume, CompactDisplay)$

$$[Assume, CompactDisplay] \quad \textbf{(109)}$$

> $Assume(0 < a)$

$$\{a::(0, \infty)\} \quad \textbf{(110)}$$

> $C_2 := x^2 + y^2 + z^2 = a^2$

$$C_2 := x^2 + y^2 + z^2 = a^2 \quad \textbf{(111)}$$

> $ParametrizeSurface(C_2)$

$$[x(\theta, \phi) = a\sin(\theta)\cos(\phi), y(\theta, \phi) = a\sin(\theta)\sin(\phi), z(\theta, \phi) = a\cos(\theta)], [\theta, \phi] \quad \textbf{(112)}$$

From the definition of the vectorial surface element as

$$d\vec{S} = \vec{r}_u \times \vec{r}_v\,du\,dv$$

$d\vec{S}$ is a vector perpendicular to the surface of the sphere. Hence, the vectorial surface integral of $d\vec{S}$ over

the upper half of the sphere should be a vector perpendicular to the plane (x, y) in the $\hat{k}$ direction. Also, checking the form of the parametrization returned by ParametrizeSurface (above) is useful to understand what the ranges for the parameters θ and ϕ need to be in order to represent the desired region; for the upper half we have

> $(Int = int)\left(1, S\_, surface = C_2, parameters = \left[\theta = 0 .. \frac{\pi}{2}, \phi = 0 .. 2\,\pi\right]\right)$

$$\left(\int 1\ d\vec{S}\right)_{surface = (x^2 + y^2 + z^2 = a^2),\ parameters = \left[\theta = 0 .. \frac{\pi}{2},\ \phi = 0 .. 2\,\pi\right]} = a^2\,\hat{k}\,\pi \qquad \textbf{(113)}$$

From symmetry considerations, the integral of $d\vec{S}$ over the lower half of the sphere should have the same magnitude but opposite direction $(-\hat{k})$, from where the integral over the whole sphere, so for $\theta = 0 .. \pi$, should be equal to 0

> $(Int = int)\left(1, S\_, surface = C_2, parameters = [\theta = 0 .. \pi, \phi = 0 .. 2\,\pi]\right)$

$$\left(\int 1\ d\vec{S}\right)_{surface = (x^2 + y^2 + z^2 = a^2),\ parameters = [\theta = 0 .. \pi,\ \phi = 0 .. 2\,\pi]} = 0 \qquad \textbf{(114)}$$

From this example we see that to get the area computing the integral of $d\vec{S}$ over the whole surface it is necessary to take as integrand the modulus of $d\vec{S}$, that is its scalar product with a unit vector parallel to it. By definition of surface element (see after **(112)**) that unit vector is given by

> $CompactDisplay(r\_(\theta, \phi))$

$$\vec{r}(\theta, \phi)\ \textit{will now be displayed as}\ \vec{r} \qquad \textbf{(115)}$$

> $n\_ := \frac{(diff(r\_(\theta, \phi), \theta))\ \&x\ (diff(r\_(\theta, \phi), \phi))}{Norm((diff(r\_(\theta, \phi), \theta))\ \&x\ (diff(r\_(\theta, \phi), \phi)))}$

$$\vec{n} := \frac{\left(\vec{r}_\theta\right) \times \left(\vec{r}_\phi\right)}{\left\| \left(\vec{r}_\theta\right) \times \left(\vec{r}_\phi\right) \right\|} \qquad \textbf{(116)}$$

where $\vec{r}$ is the vectorial form of the parametric equations of the surface. It is easy to see this vector is the radial unit vector $\hat{r}$. For that purpose you can use the option **output = vector** of ParametrizeSurface to get the vectorial form of the parametric equations for C_2

> $ParametrizeSurface(C_2, output = vector)$

$$a \sin(\theta) \cos(\phi)\ \hat{i} + a \sin(\theta) \sin(\phi)\ \hat{j} + a \cos(\theta)\ \hat{k}, [\theta, \phi] \qquad \textbf{(117)}$$

Introduce this value of $\vec{r}$ into the expression for $\vec{n}$ and change basis to *spherical*

> $n\_ := eval(n\_, r\_(\theta, \phi) = \textbf{(117)}[1])$

(118)

$$\vec{n} := \frac{a^2 \sin(\theta)^2 \cos(\phi)\, \hat{i} + a^2 \sin(\theta)^2 \sin(\phi)\, \hat{j} + \hat{k} \sin(\theta) \cos(\theta)\, a^2}{\sqrt{a^4 \sin(\theta)^4 \cos(\phi)^2 + a^4 \sin(\theta)^4 \sin(\phi)^2 + \sin(\theta)^2 \cos(\theta)^2 a^4}} \quad (118)$$

> $n\_ := ChangeBasis(n\_, spherical)$

$$\vec{n} := \hat{r} \quad (119)$$

Putting all together, the area of a sphere of radius a is given by this closed surface integral

> $(Int = int)\left(n\_, S\_, surface = C_2, parameters = [\theta = 0..\pi, \phi = 0..2\,\pi]\right)$

$$\left(\int \hat{r} \cdot d\vec{S}\right)_{surface = (x^2 + y^2 + z^2 = a^2),\ parameters = [\theta = 0..\pi,\ \phi = 0..2\,\pi]} = 4\,a^2\,\pi \quad (120)$$

In the case of the volume of a sphere, an algebraic representation of the region is

> $C_3 := x^2 + y^2 + z^2 = r^2$

$$C_3 := x^2 + y^2 + z^2 = r^2 \quad (121)$$

from where the volume of a sphere of radius a is equal to the integral of the corresponding $dV = \vec{r}_u \cdot \left(\vec{r}_v \times \vec{r}_w\right) du\, dv\, dw$ with parameters $r = 0..a$, $\theta = 0..\pi$ and $\phi = 0..2\,\pi$

> $(Int = int)\left(1, V, volume = C_3, parameters = [r = 0..a, \theta = 0..\pi, \phi = 0..2\,\pi]\right)$

$$\left(\int 1\, dV\right)_{volume = (x^2 + y^2 + z^2 = r^2),\ parameters = [r = 0..a,\ \theta = 0..\pi,\ \phi = 0..2\,\pi]} = \frac{4\,a^3\,\pi}{3} \quad (122)$$

Using ParametrizeVolume one can also see that the vectorial representation of this region is just the position vector $\vec{r}$ written in spherical coordinates and basis

> $ParametrizeVolume(C_3, basis = spherical, output = vector)$

$$r\,\hat{r},\ [r, \theta, \phi] \quad (123)$$

Parametrization of curves, surfaces and volumes

Consider the following C representing a curve in space

> $C := \{y = x^2, z = 0\}$

$$C := \{y = x^2, z = 0\} \quad (124)$$

The parametric equations for this curve are

> $ParametrizeCurve(C)$

$$[x(t) = t, y(t) = t^2, z(t) = 0], t \quad (125)$$

The right-hand sides of equations above are the components of the position vector $\vec{r}$ in cartesian coordinates, from where a vectorial form of these equations is

> $ParametrizeCurve(C, output = vector)$

$$t^2\,\hat{j} + t\,\hat{i},\ t \tag{126}$$

The curve C can also be passed in vector form

> $C := x\_i + x^2\_j$

$$C := x^2\,\hat{j} + x\,\hat{i} \tag{127}$$

> $ParametrizeCurve(C)$

$$[x(t) = t, y(t) = t^2, z(t) = 0],\ t \tag{128}$$

The equations of circle of radius a on the (x, y) plane can be written as

> $C := \{z = 0, x^2 + y^2 = a^2\}$

$$C := \{z = 0, x^2 + y^2 = a^2\} \tag{129}$$

> $ParametrizeCurve(C)$

$$[x(\phi) = a\cos(\phi), y(\phi) = a\sin(\phi), z(\phi) = 0],\ \phi \tag{130}$$

In vector notation,

> $ParametrizeCurve(C, output = vector)$

$$a\cos(\phi)\,\hat{i} + a\sin(\phi)\,\hat{j},\ \phi \tag{131}$$

Changing the basis of this vector, for example to cylindrical, we get

> $ChangeBasis(\ \textbf{(131)}\ [1],\ cylindrical)$

$$a\,\hat{\rho} \tag{132}$$

The same result can be obtained by specifying the basis

> $ParametrizeCurve(C, output = vector, basis = cylindrical)$

$$a\,\hat{\rho},\ \phi \tag{133}$$

ParametrizeSurface and ParametrizeVolume work in the same way as ParametrizeCurve, only with two and three parameters respectively.

> $C_2 := x^2 + y^2 + z^2 = a^2$

$$C_2 := x^2 + y^2 + z^2 = a^2 \tag{134}$$

> $ParametrizeSurface(C_2)$

$$[x(\theta, \phi) = a\sin(\theta)\cos(\phi), y(\theta, \phi) = a\sin(\theta)\sin(\phi), z(\theta, \phi) = a\cos(\theta)],\ [\theta, \phi] \tag{135}$$

> $C_3 := x^2 + y^2 + z^2 = r^2$

$$C_3 := x^2 + y^2 + z^2 = r^2 \tag{136}$$

> $ParametrizeVolume(C_3)$

(137)

$$[x(r,\theta,\phi)=r\sin(\theta)\cos(\phi), y(r,\theta,\phi)=r\sin(\theta)\sin(\phi), z(r,\theta,\phi)=r\cos(\theta)], [r,\theta,\phi] \quad (137)$$

> $ParametrizeVolume(C_3, basis=spherical, output=vector)$

$$r\,\hat{r}, [r,\theta,\phi] \quad (138)$$

> $ParametrizeVolume(C_3, basis=cylindrical, output=vector)$

$$\hat{k}\,z+\rho\,\hat{\rho}, [\rho,\phi,z] \quad (139)$$

Functional Differentiation in General Relativity

Functional differentiation is a key operation in theoretical physics, used thoroughly to compute field equations using an action principle, from classical mechanics, to general relativity and quantum field theory. Maple 2022 brings a significant optimization of the related *Fundiff* command, as well as major improvements in its capabilities to handle the traditional General Relativity tensors, including the determinant of the spacetime metric.

> $restart; with(Physics)$:

For illustration, set any non-flat metric, e.g. Tolman's

> $g\_[tol]$

Systems of spacetime coordinates are: $\{X=(r,\theta,\phi,t)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X=(r,\theta,\phi,t)\}$

Setting lowercaselatin_is letters to represent space indices

The Tolman metric in coordinates $[r,\theta,\phi,t]$

Parameters: $[R(t,r), E(r)]$

Signature: $(-\,-\,-\,+)$

$$g_{\mu,\nu}=\begin{bmatrix} -\dfrac{\left(\frac{\partial}{\partial r}R(t,r)\right)^2}{1+2E(r)} & 0 & 0 & 0 \\ 0 & -R(t,r)^2 & 0 & 0 \\ 0 & 0 & -R(t,r)^2\sin(\theta)^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (140)$$

For readability, avoid redundant display of functionality and set an additional system of coordinates to perform functional differentiation

> $CompactDisplay((140))$

E(r) will now be displayed as E

(141)

$R(t, r)$ *will now be displayed as* R **(141)**

> *Coordinates*(Y)

Systems of spacetime coordinates are: $\{X = (r, \theta, \phi, t),\ Y = (y1, y2, y3, y4)\}$

$\{X, Y\}$ **(142)**

The functional derivative of the metric with regards to itself

> $g_x, g_y := g\_[\mu, \nu], g\_[\sim alpha, \sim beta]$

$$g_x, g_y := g_{\mu, \nu}, g^{\alpha, \beta} \quad \textbf{(143)}$$

> (*%Fundiff* = *Fundiff*) $\left(g_x, g_y(Y)\right)$

$$\left(\frac{\delta}{\delta g^{\alpha, \beta}(Y)}\right) g_{\mu, \nu} = \delta^{(4)}([r - y1, \theta - y2, \phi - y3, t - y4])\left(-\frac{g_{\alpha, \mu}(Y)\, g_{\beta, \nu}(Y)}{2} - \frac{g_{\alpha, \nu}(Y)\, g_{\beta, \mu}(Y)}{2}\right) \quad \textbf{(144)}$$

from where the functional derivative of its integral is the symmetric expression

> (*%Fundiff* = *Fundiff*) $\left(Intc\left(g_x, X\right), g_y\right)$

$$\left(\frac{\delta}{\delta g^{\alpha, \beta}}\right)\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g_{\mu, \nu}\, dr\right) d\theta\right) d\phi\right) dt = -\frac{g_{\alpha, \mu}\, g_{\beta, \nu}}{2} - \frac{g_{\alpha, \nu}\, g_{\beta, \mu}}{2} \quad \textbf{(145)}$$

In Maple, to represent the determinant of the metric without actually computing it you can use the inert form of it

> $g_{det} := \%g\_[determinant]$

$$g_{det} := |\, g\, | \quad \textbf{(146)}$$

The derivative of the determinant with respect to the metric itself

> (*%diff* = *diff*) $\left(g_{det}, g_y\right)$;

$$\frac{\partial}{\partial g^{\alpha, \beta}}\, |\, g\, | = -g_{\alpha, \beta}\, |\, g\, | \quad \textbf{(147)}$$

from where

> (*%Fundiff* = *Fundiff*) $\left(g_{det}, g_y(Y)\right)$;

$$\left(\frac{\delta}{\delta g^{\alpha, \beta}(Y)}\right) |\, g\, | = -g_{\alpha, \beta}(Y)\, |\, g\, |(Y)\, \delta^{(4)}([r - y1, \theta - y2, \phi - y3, t - y4]) \quad \textbf{(148)}$$

> (*%Fundiff* = *Fundiff*) $\left(Intc\left(g_{det}, X\right), g_y\right)$

$$\left(\frac{\delta}{\delta g^{\alpha,\beta}}\right)\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|g|\,dr\right)d\theta\right)d\phi\right)dt=-g_{\alpha,\beta}|g| \tag{149}$$

When formulating General Relativity using an action principle, the expression that enters the integrand in the action is the square root of the determinant

> $g_{s\_det} := \mathrm{sqrt}(-\%g\_[determinant])$

$$g_{s\_det} := \sqrt{-|g|} \tag{150}$$

Its derivative, functional derivative, and the functional derivative of its integral are now all computable

> $(\%diff = diff)\left(g_{s\_det}, g_y\right);$

$$\frac{\partial}{\partial g^{\alpha,\beta}}\left(\sqrt{-|g|}\right)=-\frac{\sqrt{-|g|}\,g_{\alpha,\beta}}{2} \tag{151}$$

> $(\%Fundiff = Fundiff)\left(g_{s\_det}, g_y(Y)\right);$

$$\left(\frac{\delta}{\delta g^{\alpha,\beta}(Y)}\right)\left(\sqrt{-|g|}\right)=-\frac{\sqrt{-|g|(Y)}\,g_{\alpha,\beta}(Y)\,\delta^{(4)}([r-y1,\theta-y2,\phi-y3,t-y4])}{2} \tag{152}$$

> $(\%Fundiff = Fundiff)\left(Intc\left(g_{s\_det}, X\right), g_y\right)$

$$\left(\frac{\delta}{\delta g^{\alpha,\beta}}\right)\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\sqrt{-|g|}\,dr\right)d\theta\right)d\phi\right)dt=\frac{g_{\alpha,\beta}|g|}{2\sqrt{-|g|}} \tag{153}$$

Likewise, for the derivative of the metric we now have

> $dg := d\_[\mathrm{rho}](g\_[\mathrm{mu},\mathrm{nu}])$

$$dg := \partial_{\rho}\left(g_{\mu,\nu}\right) \tag{154}$$

> $(\%Fundiff = Fundiff)\left(dg, g_y(Y)\right)$

$$\left(\frac{\delta}{\delta g^{\alpha,\beta}(Y)}\right)\partial_{\rho}\left(g_{\mu,\nu}\right)=\partial_{\rho}\left(\delta^{(4)}([r-y1,\theta-y2,\phi-y3,t-y4])\right)\left(-\frac{g_{\alpha,\mu}(Y)\,g_{\beta,\nu}(Y)}{2}-\frac{g_{\alpha,\nu}(Y)\,g_{\beta,\mu}(Y)}{2}\right) \tag{155}$$

from where the functional derivative of the integral of the product of dg with the square root of the determinant becomes

> $(\%Fundiff = Fundiff)\left(Intc\left(dg\cdot g_{s\_det}, X\right), g_y(Y)\right)$

(156)

$$\left(\frac{\delta}{\delta g^{\alpha,\beta}(Y)}\right)\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\partial_{\rho}\left(g_{\mu,\nu}\right)\sqrt{-|g|}\,dr\right)d\theta\right)d\phi\right)dt \tag{156}$$

$$=\frac{\partial_{\rho}\left(g_{\mu,\nu}(Y),[Y]\right)g_{\alpha,\beta}(Y)\,|g|(Y)}{2\sqrt{-|g|(Y)}}$$

$$+\frac{\partial_{\rho}(|g|(Y),[Y])\left(-\frac{g_{\alpha,\mu}(Y)\,g_{\beta,\nu}(Y)}{2}-\frac{g_{\alpha,\nu}(Y)\,g_{\beta,\mu}(Y)}{2}\right)}{2\sqrt{-|g|(Y)}}$$

The handling of this abstract representation $|g|$ of the determinant of the metric is also consistent with the expansion of the determinant both with regards to its components or its tensorial representation. Due to the amount of indices involved, it is simpler to input the expressions using lowercase latin indices, and to verify formulas set the most arbitrary possible metric

> *Setup*(*redo*, *spacetimeindices* = *lowercaselatin*, *metric* = *arbitrary*, *coordinates* = [:-*X*,:-*Y*], *quiet*) :

The metric now has the most general form in terms of ten arbitrary functions

> *g_*[ ]

$$g_{a,b}=\begin{bmatrix} \_F1(X) & \_F2(X) & \_F3(X) & \_F4(X)\\ \_F2(X) & \_F5(X) & \_F6(X) & \_F7(X)\\ \_F3(X) & \_F6(X) & \_F8(X) & \_F9(X)\\ \_F4(X) & \_F7(X) & \_F9(X) & \_F10(X)\end{bmatrix} \tag{157}$$

The tensorial form of the determinant is

> $det_g := LeviCivita[\sim p, \sim r, \sim s, \sim t]\cdot g\_[i,p]\cdot g\_[k,r]\cdot g\_[l,s]\cdot g\_[m,t]\cdot LeviCivita[\sim i, \sim k, \sim l, \sim m]$ /24

$$det_g := \frac{\epsilon^{p,r,s,t}\,g_{i,p}\,g_{k,r}\,g_{l,s}\,g_{m,t}\,\epsilon^{i,k,l,m}}{24} \tag{158}$$

This expression is now understood by the simplifier

> $Simplify(det_g)$

$$|g| \tag{159}$$

The expanded form of the determinant is given by the active form of $|g| \equiv \%g\_[determinant]$, which is entered without the `%` prefix

> *g_*[*determinant*]

$$\left(-\_F8(X)\,\_F10(X)+\_F9(X)^2\right)\_F2(X)^2+((2\,\_F6(X)\,\_F10(X) \\ -2\,\_F7(X)\,\_F9(X))\,\_F3(X)-2\,\_F4(X)\,(\_F6(X)\,\_F9(X)-\_F7(X)\,\_F8(X))) \tag{160}$$

$$\_F2(X) + \left(\_F7(X)^2 - \_F10(X)\ \_F5(X)\right)\ \_F3(X)^2 - 2\ \_F4(X)\ (\_F6(X)\ \_F7(X) - \_F9(X)\ \_F5(X))\ \_F3(X) + \left(\_F6(X)^2 - \_F8(X)\ \_F5(X)\right)\ \_F4(X)^2 - \left(\_F10(X)\ \_F6(X)^2 - 2\ \_F9(X)\ \_F6(X)\ \_F7(X) + \_F8(X)\ \_F7(X)^2 - \_F5(X)\ \left(\_F8(X)\ \_F10(X) - \_F9(X)^2\right)\right)\ \_F1(X)$$

Summing over the repeated indices of the tensorial form **(158)** and comparing with the expanded form we see they are equal

> $normal\left(SumOverRepeatedIndices\left(det_g\right) - g\_[determinant]\right)$

$$0 \qquad \textbf{(161)}$$

Depending on the case, more subtle expressions can also be mapped into $|\,g\,|$. For example,

> $LeviCivita[\sim i, \sim k, \sim l, \sim m]\cdot g\_[r, \sim b]\cdot g\_[l, s]\cdot g\_[m, t]\cdot g\_[k, \sim a]\cdot d\_[h](g\_[i, p])\cdot LeviCivita[\sim p, \sim r, \sim s, \sim t]$

$$\epsilon^{i,k,l,m}\ \delta_r^{\ b}\ g_{l,s}\ g_{m,t}\ \delta_k^{\ a}\ \partial_h\left(g_{i,p}\right)\ \epsilon^{p,r,s,t} \qquad \textbf{(162)}$$

> $Simplify(\ \textbf{(162)}\ )$

$$\left(2\ \partial_h\left(g_{i,p}\right)\ g^{a,b}\ g^{i,p} - 2\ \partial_h\left(g_{i,p}\right)\ g^{a,i}\ g^{b,p}\right)\ |\,g\,| \qquad \textbf{(163)}$$

> $LeviCivita[\sim i, \sim k, \sim l, \sim m]\cdot((g\_[t, \sim b]*g\_[l, s]\cdot g\_[m, \sim a] + g\_[s, \sim b]\cdot g\_[m, t]\cdot g\_[l, \sim a])\cdot g\_[k, r] + g\_[r, \sim b]\cdot g\_[l, s]\cdot g\_[m, t]\cdot g\_[k, \sim a])\cdot d\_[h](g\_[i, p])\cdot LeviCivita[\sim p, \sim r, \sim s, \sim t]$

$$\epsilon^{i,k,l,m}\ \left(\left(\delta_t^{\ b}\ g_{l,s}\ \delta_m^{\ a} + \delta_s^{\ b}\ g_{m,t}\ \delta_l^{\ a}\right)\ g_{k,r} + \delta_r^{\ b}\ g_{l,s}\ g_{m,t}\ \delta_k^{\ a}\right)\ \partial_h\left(g_{i,p}\right)\ \epsilon^{p,r,s,t} \qquad \textbf{(164)}$$

> $Simplify(\ \textbf{(164)}\ )$

$$\left(6\ \partial_h\left(g_{i,p}\right)\ g^{a,b}\ g^{i,p} - 6\ \partial_h\left(g_{i,p}\right)\ g^{a,i}\ g^{b,p}\right)\ |\,g\,| \qquad \textbf{(165)}$$

Likewise the simplifier can now map derivatives of the tensorial form of the determinant into the more compact abstract form

> $(\%d\_[h] = d\_[h])\left(det_g\right)$

$$\partial_h\left(\frac{\epsilon^{p,r,s,t}\ g_{i,p}\ g_{k,r}\ g_{l,s}\ g_{m,t}\ \epsilon^{i,k,l,m}}{24}\right) = \frac{1}{24}\left(\epsilon^{p,r,s,t}\ \epsilon^{i,k,l,m}\ \left(\partial_h\left(g_{i,p}\right)\ g_{k,r}\ g_{l,s}\ g_{m,t} + g_{i,p}\ \partial_h\left(g_{k,r}\right)\ g_{l,s}\ g_{m,t} + g_{i,p}\ g_{k,r}\ \partial_h\left(g_{l,s}\right)\ g_{m,t} + g_{i,p}\ g_{k,r}\ g_{l,s}\ \partial_h\left(g_{m,t}\right)\right)\right) \qquad \textbf{(166)}$$

> $Simplify(\ \textbf{(166)}\ )$

$$\partial_h\left(\frac{\epsilon_{p,r,s,t}\ g^{i,p}\ g^{k,r}\ g^{l,s}\ g^{m,t}\ \epsilon_{i,k,l,m}}{24}\right) = \partial_h\left(g_{i,k}\right)\ |\,g\,|\ g^{i,k} \qquad \textbf{(167)}$$

With the new functional differentiation and simplification capabilities in place, it is now possible to

compute more complicated abstract tensorial expressions, for example:

> $d_{det\_g} := rhs(\ \mathbf{(167)}\ )$

$$d_{det\_g} := \partial_h\left(g_{i,\,k}\right)\,|\,\mathbf{g}\,|\,g^{\,i,\,k} \tag{168}$$

> $(\%Fundiff = Fundiff)\left(d_{det\_g},\, g_{a,\,b}(Y)\right)$

$$\left(\frac{\delta}{\delta g_{a,\,b}(Y)}\right)\left(\partial_h\left(g_{i,\,k}\right)\,|\,\mathbf{g}\,|\,g^{\,i,\,k}\right) = \partial_h\left(g_{i,\,k}(X)\right)\,g^{\,i,\,k}(X)\,g^{\,a,\,b}(Y)\,|\,\mathbf{g}\,|(Y)\,\delta^{(4)}([-y1+x1,\ -y2+x2,\ -y3+x3,\ -y4+x4]) - \partial_h\left(g_{i,\,k}(X)\right)\,|\,\mathbf{g}\,|(X)\,\delta^{(4)}([-y1+x1,\ -y2+x2,\ -y3+x3,\ -y4+x4])\,g^{\,a,\,i}(Y)\,g^{\,b,\,k}(Y) + |\,\mathbf{g}\,|(X)\,g^{\,i,\,k}(X)\,\partial_h\left(\delta^{(4)}([-y1+x1,\ -y2+x2,\ -y3+x3,\ -y4+x4])\right)\delta_i^{\,a}(Y)\,\delta_k^{\,b}(Y) \tag{169}$$

It is using these new capabilities that the results **(153)** and **(156)** are computed.

In addition to handling (functional) derivatives of different representations of the determinant of the metric, in Maple 2022 *Fundiff* can compute the functional derivatives of the traditional tensors of General Relativity with respect to the metric. The key observation is that all of them can be expressed in terms of the metric itself and its derivatives. For example, for the Christoffel symbols we have

> $(\%Fundiff = Fundiff)\ (Christoffel[i,j,k],\, g\_[\sim a,\, \sim b](Y))$

$$\left(\frac{\delta}{\delta g^{\,a,\,b}(Y)}\right)\Gamma_{i,\,j,\,k} = -\frac{1}{2}\left(\partial_i\left(\delta^{(4)}([-y1+x1,\ -y2+x2,\ -y3+x3,\ -y4+x4])\right)\left(-\frac{g_{a,\,j}(Y)\,g_{b,\,k}(Y)}{2} - \frac{g_{a,\,k}(Y)\,g_{b,\,j}(Y)}{2}\right)\right) + \frac{1}{2}\left(\partial_j\left(\delta^{(4)}([-y1+x1,\ -y2+x2,\ -y3+x3,\ -y4+x4])\right)\left(-\frac{g_{a,\,i}(Y)\,g_{b,\,k}(Y)}{2} - \frac{g_{a,\,k}(Y)\,g_{b,\,i}(Y)}{2}\right)\right) + \frac{1}{2}\left(\partial_k\left(\delta^{(4)}([-y1+x1,\ -y2+x2,\ -y3+x3,\ -y4+x4])\right)\left(-\frac{g_{a,\,i}(Y)\,g_{b,\,j}(Y)}{2} - \frac{g_{a,\,j}(Y)\,g_{b,\,i}(Y)}{2}\right)\right) \tag{170}$$

This result is computed by first relating $\Gamma_{i,\,j,\,k}$ to the metric and its derivatives

> $Christoffel[i,j,k]:$
> $\%= convert(\%,\, g\_)$

$$\Gamma_{i,\,j,\,k} = \frac{\partial_k\left(g_{i,\,j}\right)}{2} + \frac{\partial_j\left(g_{i,\,k}\right)}{2} - \frac{\partial_i\left(g_{j,\,k}\right)}{2} \tag{171}$$

and then differentiation the right-hand side, to obtain the result shown in **(170)**

> (%Fundiff = Fundiff)(rhs((171)), g_[~a, ~b](Y))

$$\left(\frac{\delta}{\delta g^{a,b}(Y)}\right)\left(\frac{\partial_k(g_{i,j})}{2}+\frac{\partial_j(g_{i,k})}{2}-\frac{\partial_i(g_{j,k})}{2}\right)=-\frac{1}{2}\left(\partial_i\left(\delta^{(4)}([-y1+x1,\ -y2+x2,\ -y3+x3,\ -y4+x4])\right)\left(-\frac{g_{a,j}(Y)\,g_{b,k}(Y)}{2}-\frac{g_{a,k}(Y)\,g_{b,j}(Y)}{2}\right)\right)+\frac{1}{2}\left(\partial_j\left(\delta^{(4)}([-y1+x1,\ -y2+x2,\ -y3+x3,\ -y4+x4])\right)\left(-\frac{g_{a,i}(Y)\,g_{b,k}(Y)}{2}-\frac{g_{a,k}(Y)\,g_{b,i}(Y)}{2}\right)\right)+\frac{1}{2}\left(\partial_k\left(\delta^{(4)}([-y1+x1,\ -y2+x2,\ -y3+x3,\ -y4+x4])\right)\left(-\frac{g_{a,i}(Y)\,g_{b,j}(Y)}{2}-\frac{g_{a,j}(Y)\,g_{b,i}(Y)}{2}\right)\right) \quad (172)$$

In the same way, for instance, because *Fundiff* can now compute derivatives of $\Gamma_{i,j,k}$, through this relationship it can also compute derivatives of the *Ricci* tensor or any other one (*Einstein*, *Riemann*, *Weyl*)

> Ricci[i, j]:
  % = convert(%, Christoffel)

$$R_{i,j}=\partial_a\left(\Gamma^{a}{}_{i,j}\right)-\partial_j\left(\Gamma^{a}{}_{a,i}\right)+\Gamma^{b}{}_{i,j}\,\Gamma^{a}{}_{a,b}-\Gamma^{b}{}_{a,i}\,\Gamma^{a}{}_{b,j} \quad (173)$$

CompactDisplay and Typesetting:-Suppress unified

There are two Maple commands to avoid redundant display of functionality: Typesetting:-Suppress and Physics:-CompactDisplay which is a synonym of PDEtools:-declare. There are differences in how these commands work:

- When you PDEtools:-declare $q(t)$, the functionality is suppressed *only from the output, not from the input.* So entering q, does not result in $q(t)$.
- When you Typesetting:-Suppress $q(t)$, the functionality is suppressed *from both the output and the input.* So entering q, result in $q(t)$. In this sense, Typesetting:-Suppress works similar to the alias command.

Also,

- When you declare $q(t)$, the functionality is suppressed *also for indexed q*, so $q[j](t)$ is displayed as $q[j]$. This is particularly useful when working with indexed variables, as for example tensors are.
- When you Typesetting:-Suppress $q(t)$, the functionality *is not suppressed for indexed q*, so $q[j](t)$ *is not* displayed as $q[j]$. This makes Typesetting:-Suppress not ideal when working with indexed

variables.

Finally,

- When you PDEtools:-declare $q(t)$, in previous versions of Maple derivatives or $q(t)$ are displayed indexed, as q_t, even when the rule to display them with a dot is set (typesetdot rule, automatically set when you load Physics).
- When you Typesetting:-Suppress $q(t)$, derivatives or $q(t)$ are displayed with a dot when that rule is set.

To combine the advantages of these two commands, new in Maple 2022, when you PDEtools:-declare a function, it also automatically gets *suppressed* (i.e, a call is made to Typesetting:-Suppress in the background). So PDEtools:-declare'd functions are now also Typesetting:-Suppress'd.

The different designs were combined such that when you PDEtools:-declare a function, *things work as always* but now you also *get advantage of several things that work very well with Typesetting:-Suppress* as is the case of typesetdot.

This is the new behavior, that actually is the same as the old one but for the fixes mentioned. In what follows, recall that Physics:-CompactDisplay calls PDEtools:-declare

```
> restart; with(Physics) :
```

The following is the same as *PDEtools*:-*declare*$(q(t))$

```
> CompactDisplay(q(t))
```

$$q(t) \text{ will now be displayed as } q \qquad (174)$$

As usual, entering q results in q, not $q(t)$

```
> q
```

$$q \qquad (175)$$

```
> lprint( (175) )
q
```

The following is as usual: the functionality is omitted in the display of the output, but it is there

```
> q(t)
```

$$q \qquad (176)$$

```
> lprint( (176) )
q(t)
```

Also as usual: for declare'd functions, the functionality is also suppressed (not displayed) when the function is indexed, the typical example being any tensor function

> $q[j](t)$

$$q_j \tag{177}$$

> *lprint*(**(177)**)

```
q[j](t)
```

New: the typesetdot rule that is automatically set when you load Physics, in Maple 2022 also works with declare'd functions

> *diff* $(q(t), t)$

$$\dot{q} \tag{178}$$

> *diff* $(q[j](t), t)$

$$\dot{q}_j \tag{179}$$

In addition, the typeset got improved: the dot is placed over q, not over *q[t]*. This is relevant, for instance, when there are several indices, e.g.

> *diff* $(q[j, k, l, m](t), t)$

$$\dot{q}_{j,\,k,\,l,\,m} \tag{180}$$

New: the typesetdot rule also works with contravariant tensor indices. For example, define $q[j]$ as a tensor, and the dot does not interfere with the superscript contravariant indices

> *Setup*$(spaceindices = lowercase, tensors = q[j])$

$$\left[spaceindices = lowercaselatin,\ tensors = \left\{\gamma_{\mu},\ \sigma_{\mu},\ \partial_{\mu},\ g_{\mu,\,\nu},\ \gamma_{a,\,b},\ q_{j},\ \epsilon_{\alpha,\,\beta,\,\mu,\,\nu}\right\}\right] \tag{181}$$

> *diff* $(q[\sim j](t), t)$

$$\dot{q}^{\,j} \tag{182}$$

Copy and paste works flawlessly in all these example. E.g. copy the above and paste it:

> *diff* $\left(q_{\sim j}(t), t\right)$

$$\dot{q}^{\,j} \tag{183}$$

> *lprint*(**(183)**)

```
diff(q[~j](t),t)
```

Documentation advanced examples

One of the most important parts of the Physics project is its documentation; the illustration of the use of the package in different scenarios. The three relevant help pages for that are

- The Physics,Examples
- The Physics,Tensors
- The Physics,Updates

For Maple 2021, the first of these pages got extended with four sections: "*Vectors in Spherical Coordinates using Tensor Notation*", "*The equations of motion in curvilinear coordinates, tensor notation and Coriolis force*", "*The EnergyMomentum tensor for the Proca Lagrangian*" and "*The Gross-Pitaevskii field equations for a quantum system of identical particles*", covering new material in Vector Analysis, Mechanics and Classical Field Theory.

For Maple 2022, the first of these pages got extended with three new sections: "*Parametrization of Curves, Surfaces and Volumes*", "*Integral Vector Calculus*" and "*The StandardModel in Particle Physics*".