Physics

Maple provides a state-of-the-art environment for algebraic computations in Physics, with emphasis on ensuring that the computational experience is as natural as possible. The theme of the Physics project for Maple 2023 has been the use of the package in Education, new commands, the consolidation of the functionality introduced in previous releases, also related to the StandardModel package, and several enhancements necessary for producing the new Courseware-Support, Mechanics material.

As part of its commitment to providing the best possible computational environment in Physics, Maplesoft launched a Maple Physics: Research and Development website in 2014, which enabled users to download research versions of the package, ask questions, and provide feedback. The results from this accelerated exchange have been incorporated into the Physics package in Maple 2023. The presentation below illustrates both the novelties and the kind of mathematical formulations that can now be performed.

Courseware support: Mechanics

The Physics package, with its various subpackages, is now a resourceful, mature project. For Maple 2023, part of the focus has been in its use in Education in Physics, starting with Mechanics. The resulting material is now part of Maple's help system and can be opened to interact with it as a Maple document. The presentation is organized as a hyperlinked syllabus covering typical topics appearing in Mechanics undergrad courses of the first years. For the Maple 2023 release, Part I covers:

1 Position, velocity and acceleration in Cartesian, cylindrical and spherical coordinates

    a The position $\vec{r}(t)$ as a function of time

    b The velocity $\vec{v}(t)$

    c The acceleration $\vec{a}(t)$

    d Deriving these formulas

    e Velocity and acceleration in the case of 2-dimensional motion on the x, y plane

2 The equations of motion

    f A single particle

        i The equations of motion - vectorial form

        ii The case of constant acceleration

        iii Motion under gravitational force *close to* the Earth's surface

        iv Motion under gravitational force *not close to* the Earth's surface

New command Lagrange Equations

There is now a new Physics command, LagrangeEquations, taking advantage of the functional differentiation capabilities. LagrangeEquations receives an expression representing a Lagrangian and returns a sequence of Lagrange equations with as many equations as coordinates are indicated. The formula in the traditional case where the Lagrangian depends on 1st order derivatives of the coordinates and there is only one parameter, t, is

$$\frac{\mathrm{d}}{\mathrm{d}t} \; \frac{\partial}{\partial \vec{v}_i} \, L = \frac{\partial}{\partial \vec{r}_i} \, L$$

where $\frac{\partial}{\partial \vec{r}_i} L$ formally represents the derivative with respect to the coordinates of the ith particle, equal

to the Gradient when working in Cartesian coordinates; $\frac{\partial}{\partial \vec{v}_i} L$ represents the equivalent operation,

replacing each coordinate by the corresponding generalized velocity and $\frac{\mathrm{d}}{\mathrm{d}t}$ represents the total

derivative with respect to t, the parameter parametrizing the coordinates. In more general cases the number of parameters can be many. For example, in electrodynamics, the "coordinate" is a tensor field $A_\mu(x, y, z, t)$, there are then four coordinates, one for each of the values of the index μ, and there are four parameters (x, y, z, t). *LagrangeEquations* can handle tensors and vectors of the Physics package as well as derivatives using vectorial differential operators (see d_ and Nabla), works by performing functional differentiation (see Fundiff), and handles 1st, and higher order derivatives of the coordinates in the Lagrangian automatically.

Examples

$with(Physics) :$
$Setup(mathematicalnotation = true, coordinates = cartesian)$
$$\textit{Systems of spacetime coordinates are: } \{X = (x, y, z, t)\}$$

$$[coordinatesystems = \{X\}, mathematicalnotation = true]$$ **(1)**

The Lagrangian of a one-dimensional oscillator - small oscillations
$$L := \frac{1}{2} \, diff(x(t), t)^2 - \frac{1}{2} \, k \, x(t)^2$$

$$L := \frac{\dot{x}(t)^2}{2} - \frac{k \, x(t)^2}{2}$$ **(2)**

The corresponding Lagrange equation gives Newton's second law, a 2nd order linear ODE for $x(t)$
$LagrangeEquations(L, x)$
$$\ddot{x}(t) + x(t) \, k = 0$$ **(3)**

The Lagrangian of a pendulum of mass m and length l where the suspension point moves uniformly

over a vertical circumference centered at the origin, with a constant frequency ω

CompactDisplay$(\phi(t))$

$$\phi(t) \text{ will now be displayed as } \phi \tag{4}$$

$$L := \frac{1}{2}\, m\left(-2\, diff(\phi(t), t)\, a\, l\, \omega \sin(\omega t - \phi(t)) + diff(\phi(t), t)^2\, l^2 + 2 \cos(\phi(t))\, g\, l\right)$$

$$L := \frac{m\left(-2\, \dot{\phi}\, a\, l\, \omega \sin(\omega t - \phi) + \dot{\phi}^2\, l^2 + 2 \cos(\phi)\, g\, l\right)}{2} \tag{5}$$

The Lagrange equations

LagrangeEquations(L, ϕ)

$$-m\, l\left(-a\, \omega^2 \cos(\omega t - \phi) + \ddot{\phi}\, l + \sin(\phi)\, g\right) = 0 \tag{6}$$

The Maxwell equations can be derived as Lagrange equations as follows. For simplicity, consider Maxwell equations in vacuum. Define first a tensor representing the 4D electromagnetic field potential

Define$(A[\mu])$

$$\text{Defined objects with tensor properties}$$

$$\left\{A_\mu,\, \gamma_\mu,\, \sigma_\mu,\, \partial_\mu,\, g_{\mu,\nu},\, \epsilon_{\alpha,\beta,\mu,\nu},\, X_\mu\right\} \tag{7}$$

The electromagnetic field tensor

$F[\mu, \nu] := d\_[\mu](A[\nu](X)) - d\_[\nu](A[\mu](X))$

$$F_{\mu,\nu} := \partial_\mu(A_\nu(X)) - \partial_\nu(A_\mu(X)) \tag{8}$$

The Lagrangian

$L := F[\mu, \nu]^2$

$$L := \left(\partial_\mu(A_\nu(X)) - \partial_\nu(A_\mu(X))\right)\left(\partial^\mu\left(A^\nu(X)\right) - \partial^\nu\left(A^\mu(X)\right)\right) \tag{9}$$

Maxwell equations in 4D tensorial notation

LagrangeEquations(L, A)

$$-4\, \square\left(A^\alpha(X)\right) + 4\, \partial_\mu\left(\partial^\alpha\left(A^\mu(X)\right)\right) = 0 \tag{10}$$

New command Substitute

The command for doing syntactical exact-match substitutions is subs. Frequently, however, what we intend to accomplish with a substitution is more of a *mathematical* substitution. For *different* kinds of mathematical substitutions Maple has the commands algsubs, Physics:-SubstituteTensor, Physics:-SubstituteTensorIndices, Physics:-Library:-SubstituteOperator and Physics:-Library:-SubstituteMatrix, resulting in a myriad of commands, useful, however difficult to remember. A new single command, Physics:-Substitute, unifies all of those so that one does not need to remember which one to use.

Examples

restart;
with(*Physics*) :

• **Substitutions of sub-products and sub-sums, possibly involving noncommutative objects**

$eq := F + A\,B(x) = E[\,j\,]$

$$eq := F + A\,B(x) = E_j \tag{11}$$

$ee := A\,B(x)\,C$

$$ee := A\,B(x)\,C \tag{12}$$

$Substitute(eq, ee)$

$$-C\,\left(F - E_j\right) \tag{13}$$

• **Substitution of matrices and vectors taking mathematical equality into account**

$eq := \text{alpha} + Matrix(2, symbol = a) = \beta + Matrix(2, symbol = b)$

$$eq := \alpha + \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \beta + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \tag{14}$$

$ee := Matrix(2, symbol = a)$

$$ee := \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \tag{15}$$

$Substitute(eq, ee)$

$$-\alpha + \beta + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \tag{16}$$

• **Setup some non-commutative operands for substituion**

$Setup(quantumoperators = \{A, B, C, E, F\}) :$

Some typical examples not handled by subs, eval or algsubs
$eq := F + A\,B = E$

$$eq := F + A\,B = E \tag{17}$$

$ee := \exp(\lambda\,A\,B\,C)\,B\,\exp(-\lambda\,A\,B\,C)$

$$ee := e^{\lambda\,A\,B\,C}\,B\,e^{-\lambda\,A\,B\,C} \tag{18}$$

$Substitute(eq, ee)$

$$e^{\lambda\,(E-F)\,C}\,B\,e^{-\lambda\,(E-F)\,C} \tag{19}$$

• **Substituting tensorial sub-expressions**

$Define(A, B, C, F, G) :$

$$\textit{Defined objects with tensor properties} \tag{20}$$

$eq := A[\mu] = G[\nu, \alpha]\,A[\alpha]\,F[\mu, \nu]$

$$eq := A_\mu = A_\alpha\,F_{\mu,\nu}\,G^{\nu,\alpha} \tag{21}$$

$ee := A[v] A[v]$

$$ee := A_v A^{\ v} \tag{22}$$

Substitute(*eq, ee*)

$$A_\alpha F_{v,\beta} G^{\ \beta,\alpha} A_\kappa F^{\ v}_{\ \lambda} G^{\ \lambda,\kappa} \tag{23}$$

Setup(*spinorindices = lowercaselatin_is*)

$$[spinorindices = lowercaselatin\_is] \tag{24}$$

Define($T[\mu, i, j]$)

Defined objects with tensor properties

$$\left\{A, B, C, F, G, \gamma_\mu, \sigma_\mu, T_{\mu,i,j}, \partial_\mu, g_{\mu,v}, \epsilon_{\alpha,\beta,\mu,v}\right\} \tag{25}$$

Substitute($Dgamma[\mu][i,j] = T[\mu, i, j], Dgamma[\sim\!alpha][\sim\!k, m], disregardfreeindices$)

$$T^{\ \alpha,k}_{\quad\ m} \tag{26}$$

• **Substitutions of tensor indices**

$g\_[\alpha, \mu] A[\sim\!mu] g\_[\sim\!alpha, \sim\!nu] B[v, \sigma, \sim\!rho]$

$$g_{\alpha,\mu} g^{\ \alpha,v} A^{\ \mu} B_{v,\sigma}^{\quad\ \rho} \tag{27}$$

Check(**(27)**, *all*)

The products in the given expression check ok.

The repeated indices per term are: $[\{...\}, \{...\}, ...]$, *the free indices are:* $\{...\}$

$$[\{\alpha, \mu, v\}], \{\sigma, \sim\!rho\} \tag{28}$$

Substitute$\left(\alpha = \beta, mu = delta, sigma = rho, \textbf{(27)}\right)$

$$g_{\beta,\delta} g^{\ \beta,v} A^{\ \delta} B_{v,\rho}^{\quad\ \rho} \tag{29}$$

New command StandardModel:-Lagrangian

One of the distinctive aspects of the Standard Model is the complexity of its Lagrangian. In this context, the new StandardModel:-Lagrangian returns the *Lagrangian* of the model after symmetry breaking, optionally restricted to only the interaction terms, or only one of its QED, QCD or electroweak sectors, or only one of the different sub-terms involved in the electroweak part; all of that with the covariant derivatives, and sums over leptons and quarks optionally expanded.

The algebraic expressions returned by *Lagrangian* are fully computable, so you can use them as starting point to construct other Lagrangians (add or subtract terms), or the Action and related field equations (see d_, D_ for covariant derivatives, diff and Fundiff for functional differentiation), or to compute scattering amplitudes (see FeynmanDiagrams and FeynmanIntegral).

Examples

restart;
with(*Physics*) :

with(*StandardModel*)

Defined as the electron, muon and tau leptons and corresponding neutrinos: e_j , μ_j , τ_j , $\nu^{(e)}{}_j$, $\nu^{(\mu)}{}_j$, $\nu^{(\tau)}{}_j$

Defined as the up, charm, top, down, strange and bottom quarks: $\mathrm{u}_{A,j}$, $\mathrm{c}_{A,j}$, $\mathrm{t}_{A,j}$, $\mathrm{d}_{A,j}$, $\mathrm{s}_{A,j}$, $\mathrm{b}_{A,j}$

Defined as gauge tensors: B_μ , $\mathbb{B}_{\mu,\nu}$, A_μ , $\mathbb{F}_{\mu,\nu}$, $\mathrm{W}_{\mu,J}$, $\mathbb{W}_{\mu,\nu,J}$, $\mathrm{W}^+{}_\mu$, $\mathbb{W}^+{}_{\mu,\nu}$, $\mathrm{W}^-{}_\mu$, $\mathbb{W}^-{}_{\mu,\nu}$, Z_μ , $\mathbb{Z}_{\mu,\nu}$, $\mathrm{G}_{\mu,a}$, $\mathbb{G}_{\mu,\nu,a}$

Defined as Gell-Mann (Glambda), Pauli (Psigma) and Dirac (Dgamma) matrices: λ_a , σ_J , γ_μ

Defined as the electric, weak and strong coupling constants: g_e, g_w, g_s

Defined as the charge in units of $|g_e|$ *for 1) the electron, muon and tauon, 2) the up, charm and top, and 3) the down, strange and bottom:* $q_e = -1$, $q_u = \dfrac{2}{3}$, $q_d = -\dfrac{1}{3}$

Defined as the weak isospin for 1) the electron, muon and tauon, 2) the up, charm and top, 3) the down, strange and bottom, and 4) all the neutrinos: $I_e = -\dfrac{1}{2}$, $I_u = \dfrac{1}{2}$, $I_d = -\dfrac{1}{2}$, $I_n = \dfrac{1}{2}$

You can use the active form without the % prefix, or the 'value' command to give the corresponding value to any of the inert representations q_e, q_u, q_d, I_e, I_u, I_d, I_n

$\big[I_d, I_e, I_n, I_u, q_d, q_e, q_u,$ *BField, BFieldStrength, Bottom, CKM, Charm, Down, ElectromagneticField,* **(30)** *ElectromagneticFieldStrength, Electron, ElectronNeutrino, FSU3, Glambda, GluonField, GluonFieldStrength, HiggsBoson, Lagrangian, Muon, MuonNeutrino, Strange, Tauon, TauonNeutrino, Top, Up, WField, WFieldStrength, WMinusField, WMinusFieldStrength, WPlusField, WPlusFieldStrength, WeinbergAngle, ZField, ZFieldStrength, g_e, g_s, $g_w\big]$

The massless fields of the model are the electromagnetic and gluon fields and the three neutrinos
Setup(*massless*)

$$\big[\, masslessfields = \{ \mathbf{G}, \mathbf{\nu}^{(\mu)}, \mathbf{\nu}^{(\tau)}, \mathbf{A}, \mathbf{\nu}^{(e)} \} \,\big] \tag{31}$$

The Leptons and Quarks of the model are
StandardModel:-Leptons

$$\big[\, \mathbf{e}, \mathbf{\mu}, \mathbf{\tau}, \mathbf{\nu}^{(e)}, \mathbf{\nu}^{(\mu)}, \mathbf{\nu}^{(\tau)} \,\big] \tag{32}$$

StandardModel:-Quarks

$$\big[\, \mathbf{u}, \mathbf{c}, \mathbf{t}, \mathbf{d}, \mathbf{s}, \mathbf{b} \,\big] \tag{33}$$

The Gauge fields, and their related field strengths displayed with Open Face type fonts
StandardModel:-GaugeFields

$$\big[\, \mathbf{A}, \mathbb{F}, \mathbf{B}, \mathbb{B}, \mathbf{W}, \mathbb{W}, \mathbf{G}, \mathbb{G}, \mathbf{W}^-, \mathbb{W}^-, \mathbf{W}^+, \mathbb{W}^+, \mathbf{Z}, \mathbb{Z} \,\big] \tag{34}$$

To represent the interaction Lagrangians for the QCD and electroweak sectors as sums over leptons and quarks, all of them fermions, it is useful to introduce four anticommutative prefixes, used below as summation indices in the formulas
Setup$\big(anticommutativeprefix = \{ f_D, f_L, f_Q, f_U \} \big)$

$$\big[\, anticommutativeprefix = \{ f_D, f_L, f_Q, f_U \} \,\big] \tag{35}$$

For readability, omit from the display of formulas the functionality of all the fields entering the Standard Model (see [CompactDisplay](#)) and use the lowercase i instead of the uppercase I to represent the imaginary unit
CompactDisplay$\big(\big(StandardModel\text{:-}Leptons, StandardModel\text{:-}Quarks, StandardModel\text{:-}GaugeFields,$
$\quad HiggsBoson, f_L, f_Q, f_U, f_D \big) (X), quiet \big) :$
interface$(imaginaryunit = i) :$

The Lagrangian of the whole Standard Model after symmetry breaking, in its most compact form:
Lagrangian$(\;)$

$$\sum_{f_Q = [\mathbf{u, c, t, d, s, b}]} \overline{f_{Q_{A,j}}} \left(i \left(\gamma_\mu \right)_{j,k} \nabla_\mu - m_{f_Q} \delta_{j,k} \right) f_{Q_{A,k}} - \frac{\mathbb{G}^2_{\mu,\nu,a}}{4} - \frac{\mathbb{F}^2_{\mu,\nu}}{4} - \frac{\mathbf{W}^+_{\mu,\nu} \mathbf{W}^-_{\mu,\nu}}{2} + \tag{36}$$

$$m^2_{\mathbf{W}} \mathbf{W}^+_\mu \mathbf{W}^-_\mu - \frac{\mathbb{Z}^2_{\mu,\nu}}{4} + \frac{m^2_{\mathbf{Z}} \mathbf{Z}^2_\mu}{2} + \frac{\partial_\mu \left(\mathbf{\Phi} \right)^2}{2} - \frac{m^2_{\mathbf{\Phi}} \mathbf{\Phi}^2}{2} + \sum_{f_L = [\mathbf{e, \mu, \tau}]} \overline{f_{L_j}} \left(i \left(\gamma_\mu \right)_{j,k} \partial_\mu \left(f_{L_k} \right) \right.$$

$$\left. - m_{f_L} f_{L_j} \right) + \sum_{f_L = [\mathbf{\nu(e), \nu(\mu), \nu(\tau)}]} i \left(\gamma_\mu \right)_{j,k} \overline{f_{L_j}} \partial_\mu \left(f_{L_k} \right) + \sum_{f_Q = [\mathbf{u, c, t, d, s, b}]} \overline{f_{Q_{A,j}}} \left(i \left(\gamma_\mu \right)_{j,k} \partial_\mu \left(f_{Q_{A,k}} \right) \right.$$

$$\left. - m_{f_Q} f_{Q_{A,j}} \right) + g_e \left(\gamma_\mu \right)_{j,k} \left(q_e \sum_{f_L = [\mathbf{e, \mu, \tau}]} \overline{f_{L_j}} f_{L_k} + q_u \sum_{f_Q = [\mathbf{u, c, t}]} \overline{f_{Q_{A,j}}} f_{Q_{A,k}} + q_d \sum_{f_Q = [\mathbf{d, s, b}]} \right.$$

$$\overline{f_{Q_{A,j}}} f_{Q_{A,k}} \right) \mathbf{A}_\mu + \frac{1}{\cos(\mathbf{\theta_w})} \left(g_w \left(\left(\gamma_\mu \right)_{j,k} \left(\delta_{k,l} + \left(\gamma_5 \right)_{k,l} \right) \left(I_e \sum_{f_L = [\mathbf{e, \mu, \tau}]} \overline{f_{L_j}} f_{L_l} + I_n \sum_{f_L = [\mathbf{\nu(e), \nu(\mu), \nu(\tau)}]} \right. \right. \right.$$

$$\overline{f_{L_j}} f_{L_l} + I_u \sum_{f_Q = [\mathbf{u, c, t}]} \overline{f_{Q_{A,j}}} f_{Q_{A,l}} + I_d \sum_{f_Q = [\mathbf{d, s, b}]} \overline{f_{Q_{A,j}}} f_{Q_{A,l}} \right) - \sin\left(\mathbf{\theta_w} \right)^2 \left(\gamma_\mu \right)_{j,k} \left(q_e \sum_{f_L = [\mathbf{e, \mu, \tau}]} \overline{f_{L_j}} f_{L_k} + q_u \right.$$

$$\left(\sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \overline{f_{QA,j}} f_{QA,k} + q_d \sum_{f_Q = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \overline{f_{QA,j}} f_{QA,k}\right)\right) \mathbf{Z}_\mu\right) - \frac{1}{2}\left(g_w \sqrt{2} \left(\gamma_\mu\right)_{j,k} \left(\delta_{k,l} + \left(\gamma_5\right)_{k,l}\right)\left(\left(\right.\right.$$

$$\sum_{f_D = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \sum_{f_U = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \mathbb{M}_{f_{U'} f_D} \overline{f_{UA,j}} f_{DA,l} + \sum_{f_L = \left[[\mathbf{\nu^{(e)}}, \mathbf{e}], [\mathbf{\nu^{(\mu)}}, \mathbf{\mu}], [\mathbf{\nu^{(\tau)}}, \mathbf{\tau}]\right]} \overline{\left(f_{L_1}\right)_j(X)} \left(f_{L_2}\right)_l(X)\right) \mathbf{W}^+{}_\mu + \left(\right.$$

$$\sum_{f_D = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \sum_{f_U = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \overline{\mathbb{M}_{f_{U'} f_D}} \overline{f_{DA,j}} f_{UA,l} + \sum_{f_L = \left[[\mathbf{\nu^{(e)}}, \mathbf{e}], [\mathbf{\nu^{(\mu)}}, \mathbf{\mu}], [\mathbf{\nu^{(\tau)}}, \mathbf{\tau}]\right]} \overline{\left(f_{L_2}\right)_j(X)} \left(f_{L_1}\right)_l(X)\right) \mathbf{W}^-{}_\mu\right)\right)$$

$$- \frac{g_w m_\Phi^2 \left(\Phi^3 + \dfrac{\Phi^4}{8 m_W}\right)}{4 m_W} + \left(\frac{g_w \Phi}{m_W} + \frac{g_w^2 \Phi^2}{4 m_W^2}\right)\left(m_W^2 \mathbf{W}^+{}_\mu \mathbf{W}^-{}_\mu + \frac{\mathbf{Z}_\mu{}^2 m_Z^2}{2}\right)$$

$$- i g_w \left(\left(\mathbb{W}^+{}_{\mu,\nu} \mathbf{W}^-{}_\mu - \mathbf{W}^+{}_\mu \mathbb{W}^-{}_{\mu,\nu}\right)\left(\mathbf{A}_\nu \sin(\theta_w) - \mathbf{Z}_\nu \cos(\theta_w)\right) + \mathbf{W}^-{}_\nu \mathbf{W}^+{}_\mu \left(\mathbb{F}_{\mu,\nu} \sin(\theta_w)\right.\right.$$

$$\left.\left. - \mathbb{Z}_{\mu,\nu} \cos(\theta_w)\right)\right) - \frac{1}{4}\left(g_w^2 \left(\left(2 \mathbf{W}^+{}_\mu \mathbf{W}^-{}_\mu + \left(\mathbf{A}_\mu \sin(\theta_w)\right.\right.\right.\right.$$

$$\left. - \mathbf{Z}_\mu \cos(\theta_w)\right)^2\right)\left(2 \mathbf{W}^+{}_\nu \mathbf{W}^-{}_\nu + \left(\mathbf{A}_\nu \sin(\theta_w) - \mathbf{Z}_\nu \cos(\theta_w)\right)^2\right) + \left(\mathbf{W}^+{}_\mu \mathbf{W}^-{}_\nu\right.$$

$$\left.\left.\left. + \mathbf{W}^+{}_\nu \mathbf{W}^-{}_\mu + \left(\mathbf{A}_\mu \sin(\theta_w) - \mathbf{Z}_\mu \cos(\theta_w)\right)\left(\mathbf{A}_\nu \sin(\theta_w) - \mathbf{Z}_\nu \cos(\theta_w)\right)\right)^2\right)\right)$$

$$- \frac{g_w \left(\displaystyle\sum_{f_L = [\mathbf{e}, \mathbf{\mu}, \mathbf{\tau}, \mathbf{\nu^{(e)}}, \mathbf{\nu^{(\mu)}}, \mathbf{\nu^{(\tau)}}]} m_{f_L} \overline{f_{L_j}} f_{L_j} + \displaystyle\sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}, \mathbf{d}, \mathbf{s}, \mathbf{b}]} m_{f_Q} \overline{f_{QA,j}} f_{QA,j}\right) \Phi}{2 m_W}$$

In the output above we see, among other things, the γ_5 [Dirac matrix](#), and the Cabibbo - Kobayashi - Maskawa matrix $\mathbb{M}$, and the tensor indices of different kinds all explicit. See [StandardModel](#) for the notational conventions used, which are standard in the literature but for a few things, like a sign in the definition of γ_5, that depend on the reference.

The Quantum Electrodynamics (QED) Lagrangian

The simplest sector of this Lagrangian **(8)** is the [QED](#) one
Lagrangian(*QED*)

$$\overline{e}_j \left(i \left(\gamma_\mu\right)_{j,k} \nabla^\mu - m_e \delta_{j,k}\right) e_k - \frac{\mathbb{F}_{\mu,\nu} \mathbb{F}^{\mu,\nu}}{4} \tag{37}$$

The *applied* form can be obtained using the [Library](#) command *ApplyProductsOfDifferentialOperators* over the output **(9)** or passing the optional argument applied
Lagrangian(*QED*, *applied*)

$$\overline{e}_j \left(i \nabla_\mu(e_k) \left(\gamma^\mu\right)_{j,k} - m_e \delta_{j,k} e_k\right) - \frac{\mathbb{F}_{\mu,\nu} \mathbb{F}^{\mu,\nu}}{4} \tag{38}$$

Only the interaction part of this Lagrangian is relevant when computing scattering amplitudes. To get that part, you can either [expand](#) the covariant derivative operator or or pass the optional keyword expanded, in which case also the trace of $\mathbb{F}_{\mu,\nu}$ gets expanded

Lagrangian(*QED*, *expanded*)

$$\mathrm{i}\, \overline{e}_j\, \partial_\mu(e_k)\, \left(\gamma^\mu\right)_{j,k} - g_e\, \overline{e}_j\, e_k\, A_\mu\, \left(\gamma^\mu\right)_{j,k} - m_e\, \overline{e}_j\, e_j \tag{39}$$

$$- \frac{\left(\partial_\mu(A_\nu) - \partial_\nu(A_\mu)\right)\left(\partial^\mu\left(A^\nu\right) - \partial^\nu\left(A^\mu\right)\right)}{4}$$

then discard the non-interaction terms
remove(*has*, **(39)**, [*d*_, *m*])

$$- \left(\gamma^\mu\right)_{j,k} g_e\, \overline{e}_j\, e_k\, A_\mu \tag{40}$$

or simpler: pass the keyword *interaction*
Lagrangian(*QED*, *interaction*)

$$- \left(\gamma^\mu\right)_{j,k} g_e\, \overline{e}_j\, e_k\, A_\mu \tag{41}$$

All the algebraic expressions returned by Lagrangian are fully computable in that further calculations can proceed starting from them. For example (see [FeynmanDiagrams](#)), this is the self-energy of the electron
FeynmanDiagrams(**(41)**, *incoming* = [*Electron*], *outgoing* = [*Electron*], *numberofloops* = 1, *diagrams*)
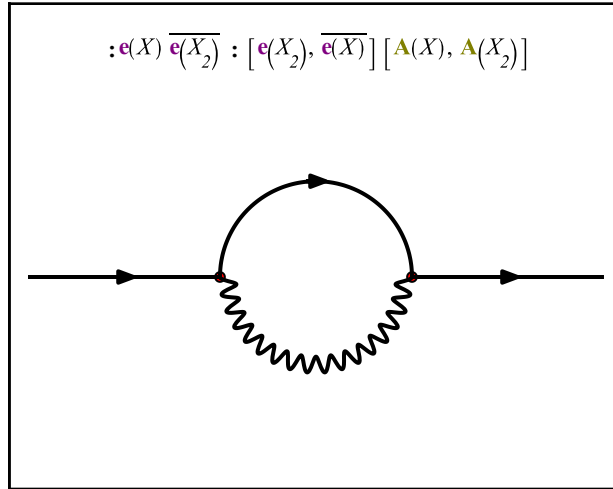
$$-\int -\frac{1}{8\,\pi^3\left(\left(P_1 + p_2\right)^2 - m_e^2 + \mathrm{i}\,\epsilon\right)\left(p_2^2 + \mathrm{i}\,\epsilon\right)}\left(\left(u_{\mathbf{e}}\right)_l\!\left(\vec{P}_1\right)\right) \tag{42}$$

$$\overline{\left(u_{\mathbf{e}}\right)_m\!\left(\vec{P}_2\right)}\, g_e^2\, \left(\gamma^\alpha\right)_{m,n}\left(\gamma^\nu\right)_{p,l}\left(\left(P_{1\beta} + p_{2\beta}\right)\left(\gamma^\beta\right)_{n,p} + m_e\,\delta_{n,p}\right) g_{\alpha,\nu}\,\delta(-P_2 + P_1)\right)\mathrm{d}p_2^{\ 4}$$

The Quantum Chromodynamics (QCD) Lagrangian

Next in complexity is the [QCD](#) Lagrangian
Lagrangian(*QCD*)

$$\sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}, \mathbf{d}, \mathbf{s}, \mathbf{b}]} \overline{f_{Q_{A,j}}}\left(\mathrm{i}\left(\gamma_\mu\right)_{j,k}\nabla^\mu - m_{f_Q}\delta_{j,k}\right)f_{Q_{A,k}} - \frac{\mathbb{G}_{\mu,\nu,a}\,\mathbb{G}^{\mu,\nu}_{a}}{4} \tag{43}$$

Lagrangian(*QCD*, *expanded*)

$$\overline{\mathbf{u}_{A,j}}\left(\mathrm{i}\left(\partial_\mu(\mathbf{u}_{A,k}) - \frac{\mathrm{i}\,g_s\,(\lambda_a)_{A,B}\,\mathbf{u}_{B,k}\,\mathbf{G}_{\mu,a}}{2}\right)(\gamma^\mu)_{j,k} - m_\mathbf{u}\,\delta_{j,k}\,\mathbf{u}_{A,k}\right) + \overline{\mathbf{c}_{A,j}}\left(\mathrm{i}\left(\partial_\mu(\mathbf{c}_{A,k})\right.\right. \quad (44)$$

$$-\frac{\mathrm{i}\,g_s\,(\lambda_a)_{A,B}\,\mathbf{c}_{B,k}\,\mathbf{G}_{\mu,a}}{2}\right)(\gamma^\mu)_{j,k} - m_\mathbf{c}\,\delta_{j,k}\,\mathbf{c}_{A,k}\right) + \overline{\mathbf{t}_{A,j}}\left(\mathrm{i}\left(\partial_\mu(\mathbf{t}_{A,k})\right.\right.$$

$$-\frac{\mathrm{i}\,g_s\,(\lambda_a)_{A,B}\,\mathbf{t}_{B,k}\,\mathbf{G}_{\mu,a}}{2}\right)(\gamma^\mu)_{j,k} - m_\mathbf{t}\,\delta_{j,k}\,\mathbf{t}_{A,k}\right) + \overline{\mathbf{d}_{A,j}}\left(\mathrm{i}\left(\partial_\mu(\mathbf{d}_{A,k})\right.\right.$$

$$-\frac{\mathrm{i}\,g_s\,(\lambda_a)_{A,B}\,\mathbf{d}_{B,k}\,\mathbf{G}_{\mu,a}}{2}\right)(\gamma^\mu)_{j,k} - m_\mathbf{d}\,\delta_{j,k}\,\mathbf{d}_{A,k}\right) + \overline{\mathbf{s}_{A,j}}\left(\mathrm{i}\left(\partial_\mu(\mathbf{s}_{A,k})\right.\right.$$

$$-\frac{\mathrm{i}\,g_s\,(\lambda_a)_{A,B}\,\mathbf{s}_{B,k}\,\mathbf{G}_{\mu,a}}{2}\right)(\gamma^\mu)_{j,k} - m_\mathbf{s}\,\delta_{j,k}\,\mathbf{s}_{A,k}\right) + \overline{\mathbf{b}_{A,j}}\left(\mathrm{i}\left(\partial_\mu(\mathbf{b}_{A,k})\right.\right.$$

$$-\frac{\mathrm{i}\,g_s\,(\lambda_a)_{A,B}\,\mathbf{b}_{B,k}\,\mathbf{G}_{\mu,a}}{2}\right)(\gamma^\mu)_{j,k} - m_\mathbf{b}\,\delta_{j,k}\,\mathbf{b}_{A,k}\right) - \frac{1}{4}\left(\left(\partial_\mu(\mathbf{G}_{\nu,a}) - \partial_\nu(\mathbf{G}_{\mu,a})\right.\right.$$

$$+ g_s f_{su3_{a,b,c}}\,\mathbf{G}_{\mu,b}\,\mathbf{G}_{\nu,c}\right)\left(\partial^\mu\left(\mathbf{G}^\nu{}_a\right) - \partial^\nu\left(\mathbf{G}^\mu{}_a\right) + g_s f_{su3_{a,d,e}}\,\mathbf{G}^\mu{}_d\,\mathbf{G}^\nu{}_e\right)\right)$$

For computing scattering amplitudes, only the interaction part of this Lagrangian is relevant

Lagrangian(*QCD, interaction*)

$$\frac{g_s\,(\lambda_a)_{A,B}\,\sum_{f_Q=[\mathbf{u,c,t,d,s,b}]}\overline{f_{Q_{A,j}}}\,f_{Q_{B,k}}\,\mathbf{G}_{\mu,a}\,(\gamma^\mu)_{j,k}}{2} - g_s f_{su3_{a,b,c}}\left(\partial_\mu(\mathbf{G}_{\nu,a})\,\mathbf{G}^\mu{}_b\,\mathbf{G}^\nu{}_c\right. \quad (45)$$

$$\left. - \frac{g_s.f_{su3_{c,d,e}}\,\mathbf{G}_{\mu,a}\,\mathbf{G}_{\alpha,b}\,\mathbf{G}^\mu{}_e\,\mathbf{G}^\alpha{}_d}{4}\right)$$

Lagrangian(*QCD, interaction, expanded*)

$$\frac{g_s\,(\lambda_a)_{A,B}\left(\overline{\mathbf{u}_{A,j}}\,\mathbf{u}_{B,k} + \overline{\mathbf{c}_{A,j}}\,\mathbf{c}_{B,k} + \overline{\mathbf{t}_{A,j}}\,\mathbf{t}_{B,k} + \overline{\mathbf{d}_{A,j}}\,\mathbf{d}_{B,k} + \overline{\mathbf{s}_{A,j}}\,\mathbf{s}_{B,k} + \overline{\mathbf{b}_{A,j}}\,\mathbf{b}_{B,k}\right)\mathbf{G}_{\mu,a}\,(\gamma^\mu)_{j,k}}{2} \quad (46)$$

$$- g_s f_{su3_{a,b,c}}\left(\partial_\mu(\mathbf{G}_{\nu,a})\,\mathbf{G}^\mu{}_b\,\mathbf{G}^\nu{}_c - \frac{g_s.f_{su3_{c,d,e}}\,\mathbf{G}_{\mu,a}\,\mathbf{G}_{\alpha,b}\,\mathbf{G}^\mu{}_e\,\mathbf{G}^\alpha{}_d}{4}\right)$$

The amplitude at tree level for the process with two incoming and two outgoing Up quarks (particle and antiparticle) exchanging a gluon

FeynmanDiagrams(**(46)**, *incomingparticles* = [*Up, conjugate*(*Up*)], *outgoingparticles* = [*Up,*
    conjugate(*Up*)], *numberofloops* = 0, *diagrams*)

$$: \overline{\mathbf{u}(X)} \, \overline{\mathbf{u}(X_2)} \, \mathbf{u}(X) \, \mathbf{u}(X_2) : \left[\mathbf{G}(X_2), \mathbf{G}(X) \right]$$
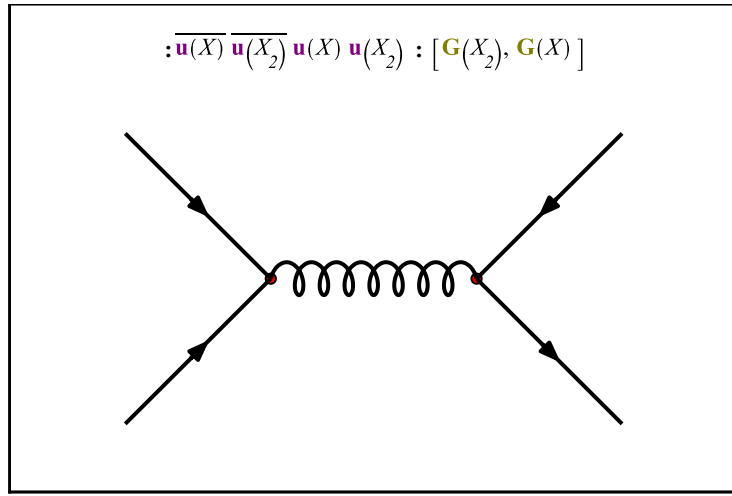
$$-\frac{1}{16\pi^2 \left((P_{1\sigma} + P_{2\sigma}) \left(P_1^{\ \sigma} + P_2^{\ \sigma} \right) + i\,\epsilon \right)} \left(i \, (\boldsymbol{u_u})_{C,l} (\vec{P_1}) \, \overline{(\boldsymbol{v_u})_{E,m}(\vec{P_2})} \right) \tag{47}$$

$$\overline{(\boldsymbol{u_u})_{F,n}(\vec{P_3})} \, (\boldsymbol{v_u})_{G,p}(\vec{P_4}) \, g_s^2 \, (\lambda_g)_{F,G} \, (\gamma^\kappa)_{n,p} \, (\lambda_f)_{E,C} \, (\gamma^\beta)_{m,l} \, g_{\beta,\kappa} \, \delta_{f,g} \, \delta\left(-P_3^{\ \lambda} - P_4^{\ \lambda} + \right.$$

$$\left. P_1^{\ \lambda} + P_2^{\ \lambda} \right) + \frac{1}{16\pi^2 \left((P_{1\sigma} - P_{3\sigma}) \left(P_1^{\ \sigma} - P_3^{\ \sigma} \right) + i\,\epsilon \right)} \left(i \, (\boldsymbol{u_u})_{C,l}(\vec{P_1}) \right.$$

$$\overline{(\boldsymbol{v_u})_{E,m}(\vec{P_2})} \, \overline{(\boldsymbol{u_u})_{F,n}(\vec{P_3})} \, (\boldsymbol{v_u})_{G,p}(\vec{P_4}) \, g_s^2 \, (\lambda_g)_{E,G} \, (\gamma^\kappa)_{m,p} \, (\lambda_f)_{F,C} \, (\gamma^\beta)_{n,l} \, g_{\beta,\kappa} \, \delta_{f,g} \, \delta\left(-P_3^{\ \lambda} \right.$$

$$\left. \left. - P_4^{\ \lambda} + P_1^{\ \lambda} + P_2^{\ \lambda} \right) \right)$$

The Electro-Weak Lagrangian

The electroweak sector of the Standard Model Lagrangian is significantly more complicated
$Lagrangian(electroweak)$

$$-\frac{\mathbb{F}_{\mu,\nu}^2}{4} - \frac{\mathbb{W}_{\mu,\nu}^+ \mathbb{W}_{\mu,\nu}^-}{2} + m_{\mathbf{W}}^2 \, \mathbf{W}_\mu^+ \, \mathbf{W}_\mu^- - \frac{\mathbb{Z}_{\mu,\nu}^2}{4} + \frac{m_{\mathbf{Z}}^2 \, \mathbf{Z}_\mu^2}{2} + \frac{\partial_\mu (\boldsymbol{\Phi})^2}{2} - \frac{m_{\boldsymbol{\Phi}}^2 \, \boldsymbol{\Phi}^2}{2} + \sum_{f_L = [\mathbf{e}, \boldsymbol{\mu}, \boldsymbol{\tau}]} \tag{48}$$

$$\overline{f_{L_j}} \left(i \, (\gamma_\mu)_{j,k} \, \partial_\mu(f_{L_k}) - m_{f_L} f_{L_j} \right) + \sum_{f_L = [\boldsymbol{\nu}^{(e)}, \boldsymbol{\nu}^{(\mu)}, \boldsymbol{\nu}^{(\tau)}]} i \, (\gamma_\mu)_{j,k} \overline{f_{L_j}} \, \partial_\mu(f_{L_k}) + \sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}, \mathbf{d}, \mathbf{s}, \mathbf{b}]}$$

$$\overline{f_{Q_A,j}} \left(i \, (\gamma_\mu)_{j,k} \, \partial_\mu(f_{Q_A,k}) - m_{f_Q} f_{Q_A,j} \right) + g_e \, (\gamma_\mu)_{j,k} \left(q_e \sum_{f_L = [\mathbf{e}, \boldsymbol{\mu}, \boldsymbol{\tau}]} \overline{f_{L_j}} f_{L_k} + q_u \sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \overline{f_{Q_A,j}} f_{Q_A,k} + q_d \right.$$

$$\left. \sum_{f_Q = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \overline{f_{Q_A,j}} f_{Q_A,k} \right) \mathbf{A}_\mu + \frac{1}{\cos(\boldsymbol{\theta_w})} \left(g_w \left((\gamma_\mu)_{j,k} \, (\delta_{k,l} + (\gamma_5)_{k,l}) \left(I_e \sum_{f_L = [\mathbf{e}, \boldsymbol{\mu}, \boldsymbol{\tau}]} \overline{f_{L_j}} f_{L_l} + I_n \right. \right. \right.$$

$$\sum_{f_L = [\boldsymbol{\nu}^{(e)}, \boldsymbol{\nu}^{(\mu)}, \boldsymbol{\nu}^{(\tau)}]} \overline{f_{L_j}} f_{L_l} + I_u \sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \overline{f_{Q_A,j}} f_{Q_A,l} + I_d \sum_{f_Q = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \overline{f_{Q_A,j}} f_{Q_A,l} \right) - \sin(\boldsymbol{\theta_w})^2 \, (\gamma_\mu)_{j,k} \left(q_e \right.$$

$$\sum_{f_L = [\mathbf{e}, \mathbf{\mu}, \mathbf{\tau}]} \overline{f_{L,j}} f_{L_k} + q_u \sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \overline{f_{Q_{A,j}}} f_{Q_{A,k}} + q_d \sum_{f_Q = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \overline{f_{Q_{A,j}}} f_{Q_{A,k}} \Bigg) \Bigg) \mathbf{Z}_\mu \Bigg)$$

$$- \frac{1}{2} \Bigg(g_w \sqrt{2} \, (\gamma_\mu)_{j,k} \left(\delta_{k,l} + (\gamma_5)_{k,l} \right) \Bigg(\Bigg(\sum_{f_D = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \sum_{f_U = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \mathbb{M}_{f_U, f_D} \overline{f_{U_{A,j}}} f_{D_{A,l}} +$$

$$\sum_{f_L = [[\mathbf{\nu}^{(\mathbf{e})}, \mathbf{e}], [\mathbf{\nu}^{(\mathbf{\mu})}, \mathbf{\mu}], [\mathbf{\nu}^{(\mathbf{\tau})}, \mathbf{\tau}]]} \overline{\left(f_{L_1} \right)_j (X)} \left(f_{L_2} \right)_l (X) \Bigg) \mathbf{W}^+_\mu + \Bigg(\sum_{f_D = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \sum_{f_U = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \overline{\mathbb{M}_{f_U, f_D}} \overline{f_{D_{A,j}}} f_{U_{A,l}} +$$

$$\sum_{f_L = [[\mathbf{\nu}^{(\mathbf{e})}, \mathbf{e}], [\mathbf{\nu}^{(\mathbf{\mu})}, \mathbf{\mu}], [\mathbf{\nu}^{(\mathbf{\tau})}, \mathbf{\tau}]]} \overline{\left(f_{L_2} \right)_j (X)} \left(f_{L_1} \right)_l (X) \Bigg) \mathbf{W}^-_\mu \Bigg) \Bigg) - \frac{g_w \, m_\Phi^2 \left(\Phi^3 + \dfrac{\Phi^4}{8 \, m_{\mathbf{w}}} \right)}{4 \, m_{\mathbf{w}}} + \Bigg(\frac{g_w \, \Phi}{m_{\mathbf{w}}}$$

$$+ \frac{g_w^2 \, \Phi^2}{4 \, m_{\mathbf{w}}^2} \Bigg) \left(m_{\mathbf{w}}^2 \, \mathbf{W}^+_\mu \, \mathbf{W}^-_\mu + \frac{\mathbf{Z}_\mu^2 \, m_{\mathbf{Z}}^2}{2} \right) - i \, g_w \left(\left(\mathbb{W}^+_{\mu, \nu} \, \mathbf{W}^-_\mu - \mathbf{W}^+_\mu \, \mathbb{W}^-_{\mu, \nu} \right) \left(\mathbf{A}_\nu \sin \left(\mathbf{\theta_w} \right) \right.$$

$$- \mathbf{Z}_\nu \cos \left(\mathbf{\theta_w} \right) \Big) + \mathbf{W}^-_\nu \, \mathbf{W}^+_\mu \left(\mathbb{F}_{\mu, \nu} \sin \left(\mathbf{\theta_w} \right) - \mathbb{Z}_{\mu, \nu} \cos \left(\mathbf{\theta_w} \right) \right) \Big) - \frac{1}{4} \left(g_w^2 \left(\left(2 \, \mathbf{W}^+_\mu \, \mathbf{W}^-_\mu \right. \right. \right.$$

$$+ \left(\mathbf{A}_\mu \sin \left(\mathbf{\theta_w} \right) - \mathbf{Z}_\mu \cos \left(\mathbf{\theta_w} \right) \right)^2 \right) \left(2 \, \mathbf{W}^+_\nu \, \mathbf{W}^-_\nu + \left(\mathbf{A}_\nu \sin \left(\mathbf{\theta_w} \right) - \mathbf{Z}_\nu \cos \left(\mathbf{\theta_w} \right) \right)^2 \right)$$

$$+ \left(\mathbf{W}^+_\mu \, \mathbf{W}^-_\nu + \mathbf{W}^+_\nu \, \mathbf{W}^-_\mu + \left(\mathbf{A}_\mu \sin \left(\mathbf{\theta_w} \right) - \mathbf{Z}_\mu \cos \left(\mathbf{\theta_w} \right) \right) \left(\mathbf{A}_\nu \sin \left(\mathbf{\theta_w} \right) - \mathbf{Z}_\nu \cos \left(\mathbf{\theta_w} \right) \right) \right)^2 \Big) \Big)$$

$$- \frac{g_w \left(\displaystyle\sum_{f_L = [\mathbf{e}, \mathbf{\mu}, \mathbf{\tau}, \mathbf{\nu}^{(\mathbf{e})}, \mathbf{\nu}^{(\mathbf{\mu})}, \mathbf{\nu}^{(\mathbf{\tau})}]} m_{f_L} \overline{f_{L_j}} f_{L_j} + \displaystyle\sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}, \mathbf{d}, \mathbf{s}, \mathbf{b}]} m_{f_Q} \overline{f_{Q_{A,j}}} f_{Q_{A,j}} \right) \Phi}{2 \, m_{\mathbf{w}}}$$

To decipher this result it is useful to see the structure of physically recognizable terms. For that purpose, you can use the keyword *showterms*

Lagrangian(*electroweak*, *showterms*)

$$L_K + L_N + L_C + L_H + L_{HV} + L_{WWV} + L_{WWVV} + L_Y \textbf{ where } \left| \, L_K = - \frac{\mathbb{F}_{\mu, \nu}^2}{4} - \frac{\mathbf{W}^+_{\mu, \nu} \, \mathbf{W}^-_{\mu, \nu}}{2} + \right. \qquad \textbf{(49)}$$

$$m_{\mathbf{w}}^2 \, \mathbf{W}^+_\mu \, \mathbf{W}^-_\mu - \frac{\mathbb{Z}_{\mu, \nu}^2}{4} + \frac{m_{\mathbf{Z}}^2 \, \mathbf{Z}_\mu^2}{2} + \frac{\partial_\mu \left(\Phi \right)^2}{2} - \frac{m_\Phi^2 \, \Phi^2}{2} + \sum_{f_L = [\mathbf{e}, \mathbf{\mu}, \mathbf{\tau}]} \overline{f_{L_j}} \left(i \, \left(\gamma_\mu \right)_{j,k} \partial_\mu \left(f_{L_k} \right) \right.$$

$$- m_{f_L} f_{L_j} \Big) + \sum_{f_L = [\mathbf{\nu}^{(\mathbf{e})}, \mathbf{\nu}^{(\mathbf{\mu})}, \mathbf{\nu}^{(\mathbf{\tau})}]} i \, \left(\gamma_\mu \right)_{j,k} \overline{f_{L_j}} \, \partial_\mu \left(f_{L_k} \right) + \sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}, \mathbf{d}, \mathbf{s}, \mathbf{b}]} \overline{f_{Q_{A,j}}} \left(i \, \left(\gamma_\mu \right)_{j,k} \partial_\mu \left(f_{Q_{A,k}} \right) \right)$$

$$- m_{f_Q} f_{Q_{A,j}} \Big), L_N = g_e \left(\gamma_\mu \right)_{j,k} \left(q_e \sum_{f_L = [\mathbf{e}, \boldsymbol{\mu}, \boldsymbol{\tau}]} \overline{f_{L_j}} f_{L_k} + q_u \sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \overline{f_{Q_{A,j}}} f_{Q_{A,k}} + q_d \sum_{f_Q = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \overline{f_{Q_{A,j}}} f_{Q_{A,k}} \right) \mathbf{A}_\mu$$

$$+ \frac{1}{\cos(\boldsymbol{\theta_w})} \left(g_w \left(\left(\gamma_\mu \right)_{j,k} \left(\delta_{k,l} + \left(\gamma_5 \right)_{k,l} \right) \left(I_e \sum_{f_L = [\mathbf{e}, \boldsymbol{\mu}, \boldsymbol{\tau}]} \overline{f_{L_j}} f_{L_l} + I_n \sum_{f_L = [\boldsymbol{\nu(\mathrm{e})}, \boldsymbol{\nu(\mu)}, \boldsymbol{\nu(\tau)}]} \overline{f_{L_j}} f_{L_l} + I_u \right. \right. \right.$$

$$\sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \overline{f_{Q_{A,j}}} f_{Q_{A,l}} + I_d \sum_{f_Q = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \overline{f_{Q_{A,j}}} f_{Q_{A,l}} \right) - \sin(\boldsymbol{\theta_w})^2 \left(\gamma_\mu \right)_{j,k} \left(q_e \sum_{f_L = [\mathbf{e}, \boldsymbol{\mu}, \boldsymbol{\tau}]} \overline{f_{L_j}} f_{L_k} + q_u \sum_{f_Q = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \right.$$

$$\overline{f_{Q_{A,j}}} f_{Q_{A,k}} + q_d \sum_{f_Q = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \overline{f_{Q_{A,j}}} f_{Q_{A,k}} \right) \right) \mathbf{Z}_\mu, L_C = -\frac{1}{2} \left(g_w \sqrt{2} \left(\gamma_\mu \right)_{j,k} \left(\delta_{k,l} + \left(\gamma_5 \right)_{k,l} \right) \left(\left(\vphantom{\sum} \right. \right. \right.$$

$$\sum_{f_D = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \sum_{f_U = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \mathbb{M}_{f_U f_D} \overline{f_{U_{A,j}}} f_{D_{A,l}} + \sum_{f_L = [[\boldsymbol{\nu(\mathrm{e})}, \mathbf{e}], [\boldsymbol{\nu(\mu)}, \boldsymbol{\mu}], [\boldsymbol{\nu(\tau)}, \boldsymbol{\tau}]]} \overline{\left(f_{L_1} \right)_j (X)} \left(f_{L_2} \right)_l (X) \right) \mathbf{W^+}_\mu + \left(\vphantom{\sum} \right.$$

$$\sum_{f_D = [\mathbf{d}, \mathbf{s}, \mathbf{b}]} \sum_{f_U = [\mathbf{u}, \mathbf{c}, \mathbf{t}]} \overline{\mathbb{M}_{f_U f_D}} \overline{f_{D_{A,j}}} f_{U_{A,l}} + \sum_{f_L = [[\boldsymbol{\nu(\mathrm{e})}, \mathbf{e}], [\boldsymbol{\nu(\mu)}, \boldsymbol{\mu}], [\boldsymbol{\nu(\tau)}, \boldsymbol{\tau}]]} \overline{\left(f_{L_2} \right)_j (X)} \left(f_{L_1} \right)_l (X) \right) \mathbf{W^-}_\mu \right), L_H$$

$$= -\frac{g_w m_\Phi^2 \left(\Phi^3 + \frac{\Phi^4}{8 m_\mathbf{w}} \right)}{4 m_\mathbf{w}}, L_{HV} = \left(\frac{g_w \Phi}{m_\mathbf{w}} + \frac{g_w^2 \Phi^2}{4 m_\mathbf{w}^2} \right) \left(m_\mathbf{w}^2 \mathbf{W^+}_\mu \mathbf{W^-}_\mu + \frac{\mathbf{Z}_\mu^2 m_\mathbf{Z}^2}{2} \right), L_{WWV}$$

$$= -\mathrm{i}\, g_w \left(\left(\mathbb{W^+}_{\mu,\nu} \mathbf{W^-}_\mu - \mathbf{W^+}_\mu \mathbb{W^-}_{\mu,\nu} \right) \left(\mathbf{A}_\nu \sin(\boldsymbol{\theta_w}) - \mathbf{Z}_\nu \cos(\boldsymbol{\theta_w}) \right) \right.$$

$$+ \mathbf{W^-}_\nu \mathbf{W^+}_\mu \left(\mathbb{F}_{\mu,\nu} \sin(\boldsymbol{\theta_w}) - \mathbb{Z}_{\mu,\nu} \cos(\boldsymbol{\theta_w}) \right) \right), L_{WWVV} = -\frac{1}{4} \left(g_w^2 \left(\left(2\, \mathbf{W^+}_\mu \mathbf{W^-}_\mu \right. \right. \right.$$

$$+ \left(\mathbf{A}_\mu \sin(\boldsymbol{\theta_w}) - \mathbf{Z}_\mu \cos(\boldsymbol{\theta_w}) \right)^2 \right) \left(2\, \mathbf{W^+}_\nu \mathbf{W^-}_\nu + \left(\mathbf{A}_\nu \sin(\boldsymbol{\theta_w}) - \mathbf{Z}_\nu \cos(\boldsymbol{\theta_w}) \right)^2 \right)$$

$$+ \left(\mathbf{W}^+{}_\mu \, \mathbf{W}^-{}_\nu + \mathbf{W}^+{}_\nu \, \mathbf{W}^-{}_\mu + \left(\mathbf{A}_\mu \sin\left(\theta_w \right) - \mathbf{Z}_\mu \cos\left(\theta_w \right) \right) \left(\mathbf{A}_\nu \sin\left(\theta_w \right) - \mathbf{Z}_\nu \cos\left(\theta_w \right) \right) \right)^2 \right) \right)$$

$$, L_Y = - \frac{g_w \left(\displaystyle\sum_{f_L = [\, \mathbf{e}, \, \mu, \, \tau, \, \nu^{(e)}, \, \nu^{(\mu)}, \, \nu^{(\tau)} \,]} m_{f_L} \overline{f_{L_j}} f_{L_j} + \displaystyle\sum_{f_Q = [\, \mathbf{u}, \, \mathbf{c}, \, \mathbf{t}, \, \mathbf{d}, \, \mathbf{s}, \, \mathbf{b} \,]} m_{f_Q} \overline{f_{Q_{A,j}}} f_{Q_{A,j}} \right) \Phi}{2 \, m_{\mathbf{W}}}$$

In this result we see a sum of L_{terms}, and after **'where'** there is a list of equations with the formulas represented by each L_{term}. Take from the above, for instance, only the *charged current L_C* term that involves interaction between the leptons and the corresponding neutrinos: you can do that with the mouse, copy and paste, or using the *term* $= ...$ option of *Lagrangian*
Lagrangian(*electroweak, term* $= L[C]$)

$$L_C = - \frac{1}{2} \left(g_w \sqrt{2} \, \left(\delta_{k,l} + \left(\gamma_5 \right)_{k,l} \right) \left(\left(\sum_{f_D = [\, \mathbf{d}, \, \mathbf{s}, \, \mathbf{b} \,]} \sum_{f_U = [\, \mathbf{u}, \, \mathbf{c}, \, \mathbf{t} \,]} \mathbb{M}_{f_U, f_D} \overline{f_{U_{A,j}}} f_{D_{A,l}} + \sum_{f_L = [\, [\nu^{(e)}, \, \mathbf{e}], \, [\nu^{(\mu)}, \, \mu], \, [\nu^{(\tau)}, \, \tau] \,]} \right. \right. \right.$$ **(50)**

$$\overline{\left(f_{L_1} \right)_j (X)} \left(f_{L_2} \right)_l (X) \right) \mathbf{W}^+{}_\mu + \left(\sum_{f_D = [\, \mathbf{d}, \, \mathbf{s}, \, \mathbf{b} \,]} \sum_{f_U = [\, \mathbf{u}, \, \mathbf{c}, \, \mathbf{t} \,]} \overline{\mathbb{M}_{f_U, f_D}} \overline{f_{D_{A,j}}} f_{U_{A,l}} + \sum_{f_L = [\, [\nu^{(e)}, \, \mathbf{e}], \, [\nu^{(\mu)}, \, \mu], \, [\nu^{(\tau)}, \, \tau] \,]}$$

$$\overline{\left(f_{L_2} \right)_j (X)} \left(f_{L_1} \right)_l (X) \right) \mathbf{W}^-{}_\mu \right) \left(\gamma^\mu \right)_{j,k} \right)$$
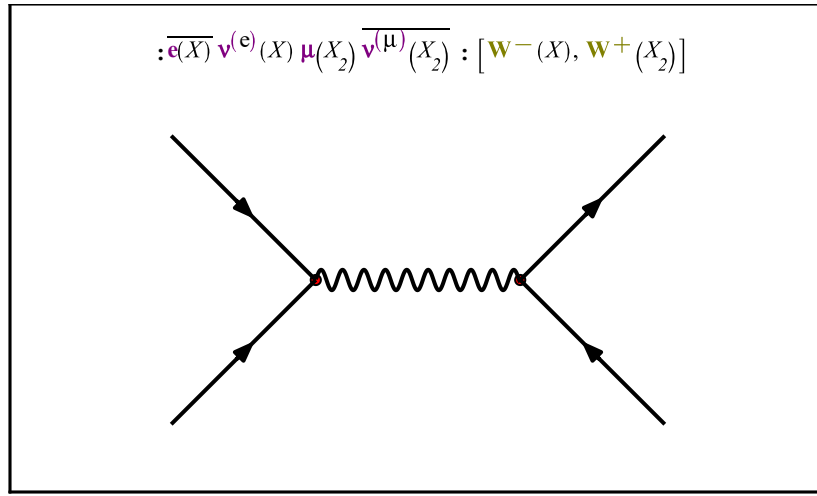
The same term in expanded form, useful for computing scattering amplitudes
Lagrangian(*electroweak, term* $= L[C]$, *expanded*)

$$L_C = - \frac{1}{2} \left(g_w \sqrt{2} \, \left(\delta_{k,l} + \left(\gamma_5 \right)_{k,l} \right) \left(\left(\mathbb{M}_{\mathbf{u},\mathbf{d}} \, \overline{\mathbf{u}_{A,j}} \, \mathbf{d}_{A,l} + \mathbb{M}_{\mathbf{c},\mathbf{d}} \, \overline{\mathbf{c}_{A,j}} \, \mathbf{d}_{A,l} + \mathbb{M}_{\mathbf{t},\mathbf{d}} \, \overline{\mathbf{t}_{A,j}} \, \mathbf{d}_{A,l} + \mathbb{M}_{\mathbf{u},\mathbf{s}} \, \overline{\mathbf{u}_{A,j}} \, \mathbf{s}_{A,l} \right. \right. \right.$$ **(51)**

$$+ \mathbb{M}_{\mathbf{c},\mathbf{s}} \, \overline{\mathbf{c}_{A,j}} \, \mathbf{s}_{A,l} + \mathbb{M}_{\mathbf{t},\mathbf{s}} \, \overline{\mathbf{t}_{A,j}} \, \mathbf{s}_{A,l} + \mathbb{M}_{\mathbf{u},\mathbf{b}} \, \overline{\mathbf{u}_{A,j}} \, \mathbf{b}_{A,l} + \mathbb{M}_{\mathbf{c},\mathbf{b}} \, \overline{\mathbf{c}_{A,j}} \, \mathbf{b}_{A,l} + \mathbb{M}_{\mathbf{t},\mathbf{b}} \, \overline{\mathbf{t}_{A,j}} \, \mathbf{b}_{A,l} + \overline{\nu^{(e)}_j} \, \mathbf{e}_l + \overline{\nu^{(\mu)}_j} \, \mu_l$$

$$+ \overline{\nu^{(\tau)}_j} \, \tau_l \right) \mathbf{W}^+{}_\mu + \left(\overline{\mathbb{M}_{\mathbf{u},\mathbf{d}}} \, \overline{\mathbf{d}_{A,j}} \, \mathbf{u}_{A,l} + \overline{\mathbb{M}_{\mathbf{c},\mathbf{d}}} \, \overline{\mathbf{d}_{A,j}} \, \mathbf{c}_{A,l} + \overline{\mathbb{M}_{\mathbf{t},\mathbf{d}}} \, \overline{\mathbf{d}_{A,j}} \, \mathbf{t}_{A,l} + \overline{\mathbb{M}_{\mathbf{u},\mathbf{s}}} \, \overline{\mathbf{s}_{A,j}} \, \mathbf{u}_{A,l} + \overline{\mathbb{M}_{\mathbf{c},\mathbf{s}}} \, \overline{\mathbf{s}_{A,j}} \, \mathbf{c}_{A,l}$$

$$+ \overline{\mathbb{M}_{\mathbf{t},\mathbf{s}}} \, \overline{\mathbf{s}_{A,j}} \, \mathbf{t}_{A,l} + \overline{\mathbb{M}_{\mathbf{u},\mathbf{b}}} \, \overline{\mathbf{b}_{A,j}} \, \mathbf{u}_{A,l} + \overline{\mathbb{M}_{\mathbf{c},\mathbf{b}}} \, \overline{\mathbf{b}_{A,j}} \, \mathbf{c}_{A,l} + \overline{\mathbb{M}_{\mathbf{t},\mathbf{b}}} \, \overline{\mathbf{b}_{A,j}} \, \mathbf{t}_{A,l} + \overline{\mathbf{e}_j} \, \nu^{(e)}_l + \overline{\mu_j} \, \nu^{(\mu)}_l + \overline{\tau_j} \, \nu^{(\tau)}_l \right)$$

$$\mathbf{W}^-{}_\mu \right) \left(\gamma^\mu \right)_{j,k} \right)$$

A process at tree level with a positron and electronic neutrino incoming and the antiparticle of the muon and a muon neutrino outgoing after exchanging a **W** boson
FeynmanDiagrams(*rhs*(**(51)**), *incoming* $= [\, conjugate(Electron), ElectronNeutrino \,]$, *outgoing* $= [\, conjugate(Muon), MuonNeutrino \,]$, *numberofloops* $= 0$, *diagrams*)

$$: \overline{e(X)} \, \nu^{(e)}(X) \, \mu(X_2) \, \overline{\nu^{(\mu)}}(X_2) : \left[W^-(X), W^+(X_2) \right]$$

$$\frac{1}{\pi^2 \left(\left(P_{1_\kappa} + P_{2_\kappa} \right) \left(P_1{}^\kappa + P_2{}^\kappa \right) - m_{W-}^2 + i\,\epsilon \right)} \left(-\frac{i}{4} \, \overline{\left(v_e \right)_m}(\vec{P_1}) \, \left(u_{\nu^{(e)}} \right)_n(\vec{P_2}) \, \left(v_\mu \right)_p(\vec{P_3}) \right. \tag{52}$$

$$\overline{\left(u_{\nu^{(\mu)}} \right)_q}(\vec{P_4}) \left(-\frac{i\sqrt{2}\, g_w \left(\gamma_5 \right)_{r,p} \left(\gamma^\alpha \right)_{q,r}}{2} - \frac{i\sqrt{2}\, g_w \left(\gamma^\alpha \right)_{q,p}}{2} \right) \Bigg($$

$$-\frac{i\sqrt{2}\, g_w \left(\gamma_5 \right)_{s,n} \left(\gamma^\nu \right)_{m,s}}{2} - \frac{i\sqrt{2}\, g_w \left(\gamma^\nu \right)_{m,n}}{2} \right) \left(-g_{\alpha,\nu} \right.$$

$$\left. \left. + \frac{\left(P_{1_\nu} + P_{2_\nu} \right) \left(P_{1_\alpha} + P_{2_\alpha} \right)}{m_{W-}^2} \right) \delta\left(-P_3{}^\beta - P_4{}^\beta + P_1{}^\beta + P_2{}^\beta \right) \right)$$
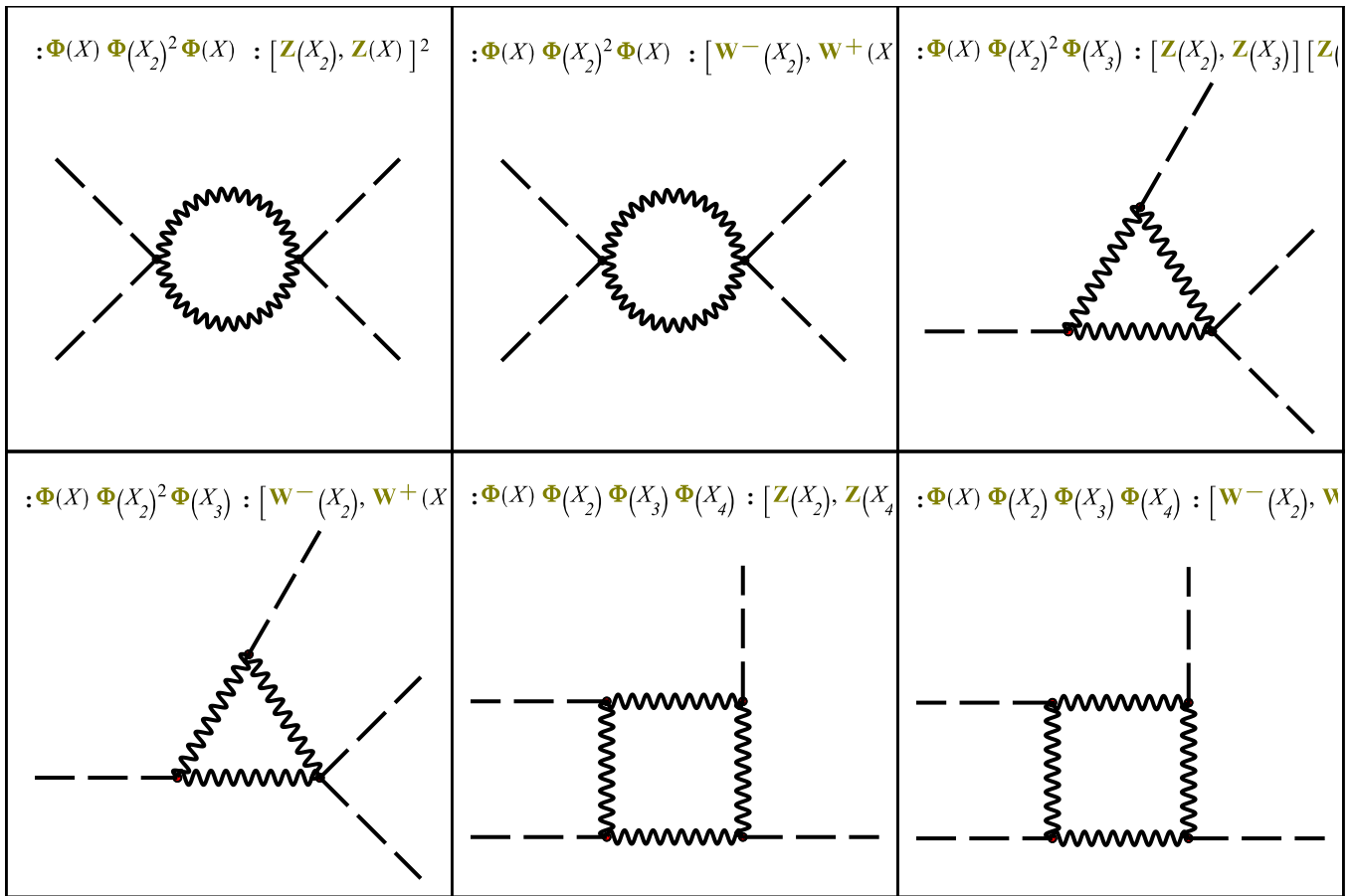
The term L_{HV} of the electroweak Lagrangian contains the interaction between the Higgs and the Z and W bosons

$Lagrangian(electroweak, term = L[HV], interaction)$

$$L_{HV} = \left(\frac{g_w \, \Phi}{m_W} + \frac{g_w^2 \, \Phi^2}{4\, m_W^2} \right) \left(m_W^2 \, W^+{}_\mu \, W^-{}^\mu + \frac{m_Z^2 \, Z_\mu \, Z^\mu}{2} \right) \tag{53}$$

The probability density at one loop for a process with three Higgs incoming and outgoing: to omit the large - two pages - algebraic result containing Feynman integrals, end the input line with ":", or remove that ending to see and manipulate the integrals. Note this result involves Feynman diagrams with two, three, four, five and six vertices. New in Maple 2023 are the diagrams with more than four vertices.

$FeynmanDiagrams(rhs((53)), incoming = [HiggsBoson, HiggsBoson, HiggsBoson], outgoing$
$\quad = [HiggsBoson, HiggsBoson, HiggsBoson], numberofloops = 1, diagrams, output$
$\quad = probabilitydensity) :$

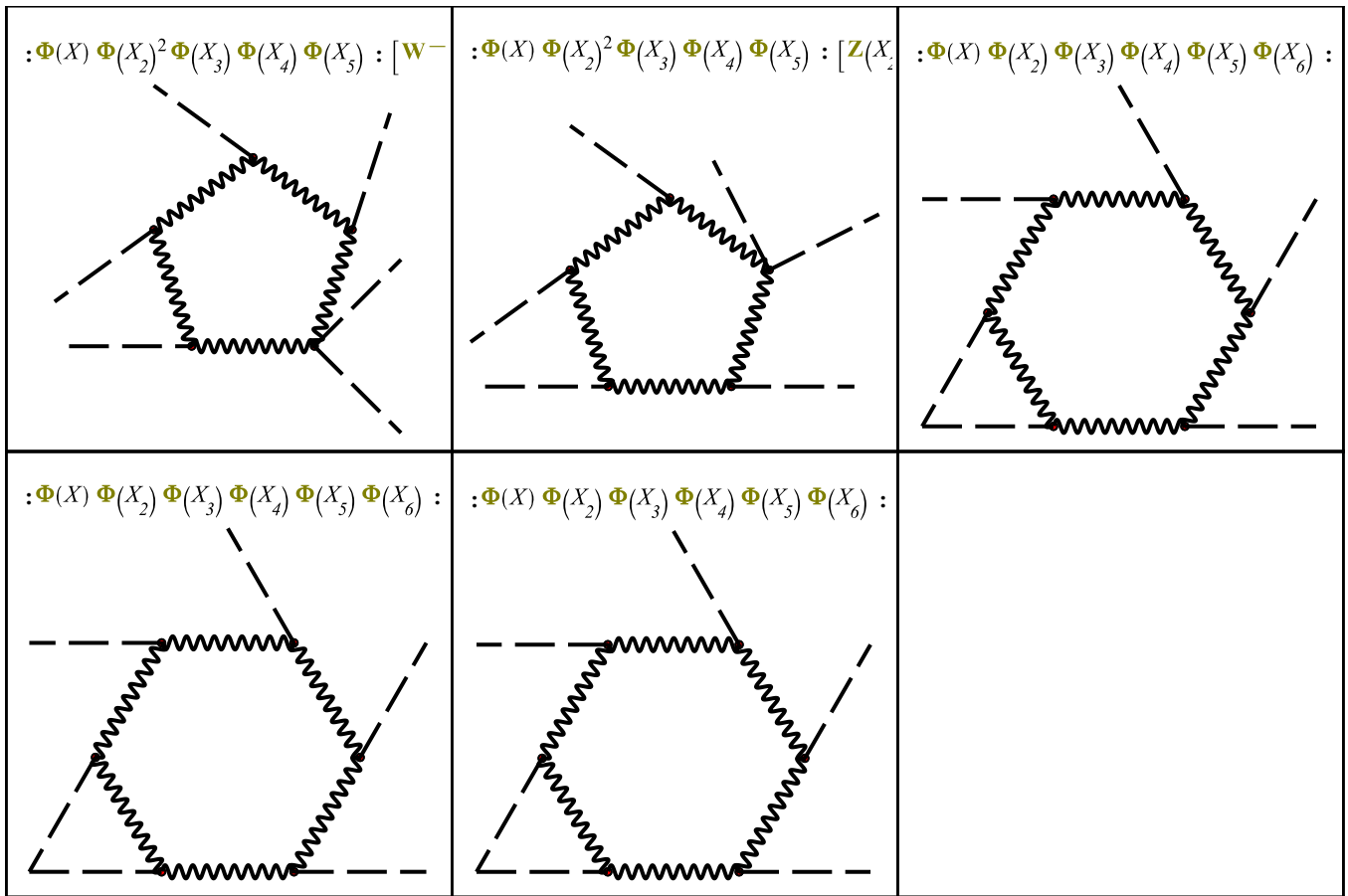$:\Phi(X)\,\Phi(X_2)^2\,\Phi(X)\;:\left[\mathbf{Z}(X_2),\mathbf{Z}(X)\right]^2$

$:\Phi(X)\,\Phi(X_2)^2\,\Phi(X)\;:\left[\mathbf{W}^-(X_2),\mathbf{W}^+(X\right.$

$:\Phi(X)\,\Phi(X_2)^2\,\Phi(X_3)\;:\left[\mathbf{Z}(X_2),\mathbf{Z}(X_3)\right]\left[\mathbf{Z}\right.$

$:\Phi(X)\,\Phi(X_2)^2\,\Phi(X_3)\;:\left[\mathbf{W}^-(X_2),\mathbf{W}^+(X\right.$

$:\Phi(X)\,\Phi(X_2)\,\Phi(X_3)\,\Phi(X_4)\;:\left[\mathbf{Z}(X_2),\mathbf{Z}(X_4\right.$

$:\Phi(X)\,\Phi(X_2)\,\Phi(X_3)\,\Phi(X_4)\;:\left[\mathbf{W}^-(X_2),\mathbf{W}\right.$

$FeynmanDiagrams(rhs(\mathbf{(53)}),\ incoming=[HiggsBoson,\ HiggsBoson,\ HiggsBoson],\ outgoing$
$=[HiggsBoson,\ HiggsBoson,\ HiggsBoson],\ numberofloops=1,\ diagrams,\ output$
$=probabilitydensity)\ :$

$:\Phi(X)\,\Phi(X_2)^2\,\Phi(X_3)^2\,\Phi(X)\;:\left[\mathbf{Z}(X_2),\mathbf{Z}(\right.$

$:\Phi(X)\,\Phi(X_2)^2\,\Phi(X_3)^2\,\Phi(X)\;:\left[\mathbf{W}^-(X_2),\right.$

$:\Phi(X)\,\Phi(X_2)^2\,\Phi(X_3)^2\,\Phi(X_4)\;:\left[\mathbf{W}^-(X_2),\right.$

$: \Phi(X) \, \Phi(X_2)^2 \, \Phi(X_3)^2 \, \Phi(X_4) : \left[Z(X_2), Z(\right.$	$: \Phi(X) \, \Phi(X_2)^2 \, \Phi(X_3)^2 \, \Phi(X_4) : \left[Z(X_2), Z(\right.$	$: \Phi(X) \, \Phi(X_2) \, \Phi(X_3) \, \Phi(X_4)^2 \, \Phi(X) : \left[Z(X_2\right.$
$: \Phi(X) \, \Phi(X_2) \, \Phi(X_3) \, \Phi(X_4)^2 \, \Phi(X) : \left[W^-\right.$	$: \Phi(X) \, \Phi(X_2)^2 \, \Phi(X_3)^2 \, \Phi(X_4) : \left[W^-(X_2),\right.$	$: \Phi(X) \, \Phi(X_2) \, \Phi(X_3) \, \Phi(X_4)^2 \, \Phi(X) : \left[W^-\right.$
$: \Phi(X) \, \Phi(X_2) \, \Phi(X_3) \, \Phi(X_4)^2 \, \Phi(X) : \left[W^-\right.$	$: \Phi(X) \, \Phi(X_2)^2 \, \Phi(X_3)^2 \, \Phi(X_4) : \left[W^-(X_2),\right.$	$: \Phi(X) \, \Phi(X_2)^2 \, \Phi(X_3) \, \Phi(X_4) \, \Phi(X_5) : \left[W^-\right.$
$: \Phi(X) \, \Phi(X_2)^2 \, \Phi(X_3) \, \Phi(X_4) \, \Phi(X_5) : \left[W^-\right.$	$: \Phi(X) \, \Phi(X_2)^2 \, \Phi(X_3) \, \Phi(X_4) \, \Phi(X_5) : \left[W^-\right.$	$: \Phi(X) \, \Phi(X_2)^2 \, \Phi(X_3) \, \Phi(X_4) \, \Phi(X_5) : \left[Z(X_2\right.$

$:\Phi(X)\,\Phi(X_2)^2\,\Phi(X_3)\,\Phi(X_4)\,\Phi(X_5):\,[\mathbf{W}^-$	$:\Phi(X)\,\Phi(X_2)^2\,\Phi(X_3)\,\Phi(X_4)\,\Phi(X_5):\,[\mathbf{Z}(X_2$	$:\Phi(X)\,\Phi(X_2)\,\Phi(X_3)\,\Phi(X_4)\,\Phi(X_5)\,\Phi(X_6):$
$:\Phi(X)\,\Phi(X_2)\,\Phi(X_3)\,\Phi(X_4)\,\Phi(X_5)\,\Phi(X_6):$	$:\Phi(X)\,\Phi(X_2)\,\Phi(X_3)\,\Phi(X_4)\,\Phi(X_5)\,\Phi(X_6):$	

New Physics:-Library commands: FactorSum and RemoveCommonFactors

It is sometimes the case that one needs to factor out something *without changing or complicating the algebraic structure of each factor.* For this purpose, there is a new *Physics:-Library* command, *FactorSum.*

Examples

restart;

$$f := A\,\sin(x)\,\theta(x)\,k - \frac{A\,\sin(x\,)\cdot\theta(x)\,m\,\omega^2}{B+C}$$

$$f := A\,\sin(x)\,\theta(x)\,k - \frac{A\,\sin(x)\,\theta(x)\,m\,\omega^2}{B+C} \tag{54}$$

We want to extract the factors $A\,\sin(x)\,\theta(x)$. The standard approach is to use *factor* or *simplify*

factor(f)

$$\frac{A\,\sin(x)\,\theta(x)\,\left(-m\,\omega^2 + k\,B + k\,C\right)}{B+C} \tag{55}$$

simplify$(f, size)$

$$\frac{A \sin(x)\, \theta(x)\, \left(k\,(B+C) - m\,\omega^2\right)}{B+C} \tag{56}$$

Both performed the operation requested, but, in addition, performed additional undesired operations or normalizations. An alternative to accomplish this *without changing the algebraic structure of each factor*, is

Physics:-*Library*:-*FactorSum*(*f*)

$$\left(k - \frac{m\,\omega^2}{B+C}\right) A \sin(x)\, \theta(x) \tag{57}$$

The relevance of this is in that more often than not, textbooks write this expression as in **(57)**, not **(56)**, not **(55)**, and when studying a book or trying to follow its computations getting *exact match* to what we see on paper, *Physics*:-*Library*:-*FactorSum* is of help in a way that the other commands are not.

Similarly, when we have equations with both sides having *a common factor* and therefore we want to just cancel out that factor *without changing anything else*, the routine to use is the new *Physics*:-*Library*:-*RemoveCommonFactors*. For example, in

(57) = $A \sin(x)\, \theta(x)$

$$\left(k - \frac{m\,\omega^2}{B+C}\right) A \sin(x)\, \theta(x) = A \sin(x)\, \theta(x) \tag{58}$$

we see, on the left-hand and right-hand sides, the common factors $A \sin(x)\, \theta(x)$. One simple way of performing this cancellation of common factors on the left and right hand sides, sufficiently identical to what we do with paper and pencil, is

Physics:-*Library*:-*RemoveCommonFactors*(**(58)**)

$$k - \frac{m\,\omega^2}{B+C} = 1 \tag{59}$$

Note that, as shown above, you don't need to load the Physics to use its Library commands.

New coordinates labels

Notation is one of the most important things to communicate with others in science. In the context of the Physics package, strong emphasis is put on using textbook notation as much as possible regarding input and output. Still, for historical reasons, when using the Physics package, the labels used to refer to a *coordinate system* had been *a single Capital Letter*, as in *X, Y, ...*It was not possible to use, e.g. X', or x.

That has changed in Maple 2023: *any symbol* can be used as a coordinate system label.

Examples

with(*Physics*) :

A lowercase letter can be used as a coordinate's label
Setup(*coordinates* = *x*)

$$\textit{Systems of spacetime coordinates are: } \{x = (x_1, x_2, x_3, x_4)\}$$

$$[\textit{coordinatesystems} = \{x\}] \tag{60}$$

In these cases where the coordinate's system label *is not a single Capital Letter, say X,* the default coordinates automatically associated to the label, instead of $[x1, x2, ...]$, are now *literal subscripts* $[x_1, x_2, ...]$, that is: symbols with a root, in this example *x,* followed by the double underscore __, then a number. The automatic mapping $x0 \rightarrow x4$ is in these cases $x_0 \rightarrow x_4$, also using literal subscripts $x\_\_0$ and $x\_\_4$.

Hence
x_0

$$x_4 \tag{61}$$

Everything else works as with the traditional Capital Letter coordinate-system labels. So x^{μ} is a tensor because it is the 4-vector related to the system of coordinates $x = (x_1, x_2, x_3, x_4)$
type(*x*[mu], *Library:-PhysicsType:-Tensor*)

$$\textit{true} \tag{62}$$

Consequently, $x[\text{mu}]^2$ also receives the automatic reformatting of repeated indices making one of then contravariant
$x[\text{mu}]^2$

$$x_{\mu} x^{\mu} \tag{63}$$

SumOverRepeatedIndices(**(63)**)

$$-x_1^{\,2} - x_2^{\,2} - x_3^{\,2} + x_4^{\,2} \tag{64}$$

A typical textbook case example: the label has a prime using the new feature of Maple 2023 that allows for *primed variables.* For example,
Coordinates(*X'*)

$$\textit{Systems of spacetime coordinates are: } \{X' = (x'_1, x'_2, x'_3, x'_4), x = (x_1, x_2, x_3, x_4)\}$$

$$\{X', x\} \tag{65}$$

Note that the coordinates automatically associated to X' also have a prime; that is also as in textbooks. One can as well use x and x' at the same time to describe transformations of coordinates between two reference systems

$Coordinates(x')$

Unaliasing $\{X'\}$ previously defined as a system of spacetime coordinates

Systems of spacetime coordinates are: $\left\{ x = \left(x_1, x_2, x_3, x_4 \right), x' = \left(x'_1, x'_2, x'_3, x'_4 \right) \right\}$

$$\{x, x'\} \tag{66}$$

(in the above, the previous label X' is automatically removed to reuse the coordinates $\left(x'_1, x'_2, x'_3, x'_4 \right)$ with the new label x')

The case of a coordinate system where the label has a *prime* is also special. For example, suppose you set a coordinate system *without a prime*, as in

$Coordinates(X = cartesian)$

Systems of spacetime coordinates are: $\left\{ X = (x, y, z, t), x = \left(x_1, x_2, x_3, x_4 \right), x' = \left(x'_1, x'_2, x'_3, x'_4 \right) \right\}$

$$\{X, x, x'\} \tag{67}$$

If you now set the label X', because the label X is already set, the coordinates associated to X' are those of X with a prime

$Coordinates(X')$

Systems of spacetime coordinates are: $\left\{ X = (x, y, z, t), X' = (x', y', z', t'), x = \left(x_1, x_2, x_3, x_4 \right), x' \right.$

$\left. = \left(x'_1, x'_2, x'_3, x'_4 \right) \right\}$

$$\{X, X', x, x'\} \tag{68}$$

Finally, it is now also possible to use *literalsubscripts* with any label. For example, for a Capital Letter label, say instead of $Y = (y1, y2, y3, y4)$, you can set Y with literal subscripts:

$Coordinates(Y, literalsubscripts)$

Systems of spacetime coordinates are: $\left\{ X = (x, y, z, t), X' = (x', y', z', t'), Y = \left(y_1, y_2, y_3, y_4 \right), x \right.$

$\left. = \left(x_1, x_2, x_3, x_4 \right), x' = \left(x'_1, x'_2, x'_3, x'_4 \right) \right\}$

$$\{X, X', Y, x, x'\} \tag{69}$$

Summarizing:

1. The label can now be any symbol; particularly useful is the case of a *lowercase* letter as a label: we can now represent position vectors in tensorial notation using a more natural textbook-notation (lowercase).
2. We can now use the traditional x and x', or X and X' notations, at the same time, as frequently seen in textbooks when presenting transformations of coordinates
3. The coordinates automatically associated to these new coordinate labels use *literal subscripted letters* as in $x_1, x_2, ...$, a textbook-like typesetting, instead of the old, not so good looking, $x1, x2, ...$
4. It is now possible (optional) to use *literal subscripts* also with Capital Letter coordinate-system labels
5. Coordinate-system labels with a *prime* are treated in a special way when the same label *without a prime* already exists, by just concatenating a prime to the coordinates of the label without prime.

Documentation advanced examples

One of the most important parts of the Physics project is its documentation; the illustration of the use of the package in different scenarios. The three relevant help pages for that are

- The Physics,Examples

- The Physics,Tensors

- The Physics,Updates

For Maple 2021, the first of these pages got extended with four sections covering new material in Vector Analysis, Mechanics and Classical Field Theory.
1. *Vectors in Spherical Coordinates using Tensor Notation*
2. *The equations of motion in curvilinear coordinates, tensor notation and Coriolis force*
3. *The EnergyMomentum tensor for the Proca Lagrangian*
4. *The Gross-Pitaevskii field equations for a quantum system of identical particles*"

For Maple 2022, the first of these pages got extended with three new sections:
1. *Parametrization of Curves, Surfaces and Volumes*
2. *Integral Vector Calculus*
3. *The StandardModel in Particle Physics*

For **Maple 2023**, the first of these pages got extended with three new sections:

1 . Moving charges: the retarded and Liénard-Wiechert potentials, electric and magnetic fields $\vec{E}$ and $\vec{H}$

2 . Deriving the mathematical form of 4D relativistic Lorentz transformations

3 . Einstein's principle of relativity